CONJECTURES OF ALPERIN AND BROUÉ FOR 2-BLOCKS WITH ELEMENTARY ABELIAN DEFECT GROUPS OF ORDER 8

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Abstract. Using the classification of finite simple groups, we prove Alperin's weight conjecture and the character theoretic version of Broué's abelian defect conjecture for 2-blocks of finite groups with an elementary abelian defect group of order 8.

1. Introduction

Throughout this paper p is a prime and $\mathcal{O}$ a complete discrete valuation ring having an algebraically closed residue field k of characteristic p and a quotient field K of characteristic 0, which is always assumed to be large enough for the finite groups under consideration. For G a finite group, a *block of* $\mathcal{O}G$ or kG is a primitive idempotent b in $Z(\mathcal{O}G)$ or $Z(kG)$, respectively. The canonical map $\mathcal{O}G \to kG$ induces a defect group preserving bijection between the sets of blocks of $\mathcal{O}G$ and kG. By Brauer's First Main Theorem there is a canonical bijection between the set of blocks of kG with a fixed defect group P and the set of blocks of $kN_G(P)$ with P as a defect group. Alperin's weight conjecture predicts that the number $\ell(b)$ of isomorphism classes of simple kGb-modules is an invariant of the local structure of b. For b a block of kG with an abelian defect group P, denoting by c the block of $kN_G(P)$ corresponding to b, Alperin's weight conjecture holds if and only if the block algebras of b and c have the same number of isomorphism classes of simple modules, or equivalently, the same number of ordinary irreducible characters. Thus, for blocks with abelian defect groups, Alperin's weight conjecture would be implied by any of the versions of Broué's Abelian Defect Conjecture, predicting that there should be a perfect isometry, or isotypy, or even a (splendid) derived equivalence between the block algebras, over $\mathcal{O}$, of b and c. Alperin announced the weight conjecture in [1]. At that time, the conjecture was known to hold for all blocks of finite groups with cyclic defect groups (by work of Brauer and Dade), dihedral, generalised quaternion, semidihedral defect groups (by work of Brauer and Olsson), and all defect groups admitting only the trivial fusion system (by the work of Broué and Puig on nilpotent blocks). Since then many authors have contributed to proving Alperin's weight conjecture for various classes of finite groups - such as finite p-solvable groups (Okuyama), finite groups of Lie type in defining characteristic (Cabanes), symmetric and general linear groups (Alperin, Fong, An) and some sporadic simple groups (An).

Theorem 1.1. *Suppose $p = 2$. Let G be a finite group and let b be a block of kG with an elementary abelian defect group P of order 8. Denote by c the block of $kN_G(P)$ corresponding to b. Then b and c have eight ordinary irreducible characters and there is an isotypy between b and c; in particular, Alperin's weight conjecture holds for all blocks of finite groups with an elementary abelian defect group P of order 8.*

For principal blocks, Theorem 1.1 follows from work of Landrock [48] and Fong and Harris [34]. Theorem 1.1 implies Alperin's weight conjecture for all blocks with a defect group of order at most 8. Indeed, the groups C_2, C_4, C_8 and $C_2 \times C_4$ admit no automorphisms of odd order, hence arise as defect groups only of nilpotent blocks, and by work of Brauer [10], [11] and Olsson [62], Alperin's weight conjecture is known in the case of $C_2 \times C_2$, D_8 and Q_8. Using a stable equivalence due to Rouquier we show in Theorem 5.1 that for blocks with an elementary abelian defect group of order 8 Alperin's weight conjecture implies Broué's isotypy conjecture.

The proof of Theorem 1.1 uses the classification of finite simple groups. By the work of Landrock already mentioned, Alperin's weight conjecture holds for blocks with an elementary abelian defect group of order 8 if and only if all irreducible characters in the block have height zero (this is the 'if' part of Brauer's height zero conjecture, which predicts that all characters in a block have height zero if and only if the defect groups are abelian). This part of Brauer's height zero conjecture has been reduced to quasi-simple finite groups by Berger and Knörr [5]; we verify in Theorem 4.1 that this reduction works within the realm of blocks with an elementary abelian defect group of order at most 8 and certain fusion patterns. We finally prove Alperin's weight conjecture for blocks with an elementary abelian defect group of order 8 for quasi-simple groups in the remaining sections. For certain classes, such as central extensions of alternating groups, sporadic groups or finite groups of Lie type defined over a field of characteristic 2, this is a simple inspection (based on calculations and well-known results by many authors) and yields results for higher rank defect groups as well:

Theorem 1.2. *Suppose $p = 2$. Let G be a quasi-simple finite group such that $Z(G)$ has odd order.*

(i) If $G/Z(G)$ is isomorphic to an alternating group A_n, $n \geq 5$, then kG has no block with an elementary abelian defect group of order 2^r, where $r \geq 3$.

(ii) If $G/Z(G)$ is a sporadic simple group or a finite group of Lie type defined over a field of characteristic 2 and if kG has a block b with an elementary abelian defect group of order 2^r, where $r \geq 3$, then either b is the principal block of $\mathrm{PSL}_2(2^r)$ or $r = 3$ and b is the principal block of J_1, or b is a non-principal block of Co_3, and Alperin's weight conjecture holds in these cases.

This follows from combining 6.3, 8.1, 8.2 and 9.1 below. Before embarking on the verification for finite groups of Lie type defined over a field of odd characteristic, we need further background material on these groups and their local structure, collected in the sections §10, §11. Assembling these parts yields the proof of 1.1 in §20. It is noticeable how few blocks of quasi-simple groups have an elementary abelian defect group - and when they do, many of them are nilpotent:

Theorem 1.3. *Suppose $p = 2$. Let q be an odd prime power and G a finite quasi-simple group. Suppose that $Z(G)$ has odd order.*

(i) If $G/Z(G)$ is a simple group of Lie type $A_n(q)$ or ${}^2A(q)$ and if b is a block of kG with an elementary abelian defect group of order 2^r for some $r \geq 3$ then r is even and b satisfies Alperin's weight conjecture.

(ii) If $G/Z(G)$ is a simple group of Lie type $B_n(q)$, $C_n(q)$, $D_n(q)$ then every block of kG with an elementary abelian defect group of order 2^r for some integer $r \geq 3$ is nilpotent.

(iii) If $G/Z(G)$ is simple of type $G_2(q)$ or ${}^3D_4(q)$ then kG has no block with an elementary abelian defect group of order 2^r, where $r \geq 3$.

(iv) If $G/Z(G)$ is simple of type ${}^2G_2(q)$ and if b is a block of kG with an elementary abelian defect group of order 2^r, where $r \geq 3$, then $r = 3$ and b is the principal block of ${}^2G_2(q)$, and Alperin's weight conjecture holds in this case.

This follows from combining Corollary 10.2, the Theorems 12.1, 13.1, 14.1 and the Propositions 15.1, 15.2, 15.3 below. Note the absence of the exceptional types F and E in the above result - we do not know whether these groups actually have blocks with elementary abelian defect groups of order 8. What we show is that if they do then these blocks satisfy Alperin's weight conjecture. At present there seems to be no general reduction for blocks with elementary abelian defect groups of order 2^r for $r \geq 4$, but the above results for quasi-simple groups would allow us to settle Alperin's weight conjecture in infinitely many cases if we did have a satisfactory reduction. More precisely, using the above results and the fact (see [22, Corollary]) that the 2-rank of exceptional finite simple groups of Lie type in odd characteristic is at most 9 we obtain the following.

Corollary 1.4. *Suppose $p = 2$. Let be G a finite quasi-simple group such that $Z(G)$ has odd order. Every block of kG with an elementary abelian defect group of order 2^r for some integer $r \geq 10$ satisfies Alperin's weight conjecture.*

2. Reduction techniques

For general background on block theory we refer to [73]. Given a finite group G and a block b of $\mathcal{O}G$ or of kG, we denote by $\mathrm{Irr}_K(G, b)$ the set of ordinary irreducible K-valued characters of G associated with b and by $\mathrm{Irr}_k(G, b)$ the set of irreducible Brauer characters of G associated with b. We set $\ell(b) = |\mathrm{Irr}_k(G, b)|$. We denote by $\mathbb{Z}\mathrm{Irr}_K(G, b)$ the group of class functions on G generated by $\mathrm{Irr}_K(G, b)$, and by $\mathbb{Z}\mathrm{Irr}_k(G, b)$ the corresponding group of class functions on the set of p-regular elements in G. By a classical result of Brauer, the *decomposition map* $\mathbb{Z}\mathrm{Irr}_K(G, b) \to \mathbb{Z}\mathrm{Irr}_k(G, b)$ induced by restriction of class functions to p-regular elements is surjective. The kernel of the decomposition map, denoted by $L^0(G, b)$, consists of all class functions associated with b which vanish on the set of p-regular elements of G, or equivalently, all generalised characters in $\mathbb{Z}\mathrm{Irr}_K(G, b)$ which are perpendicular to the characters of the projective indecomposable $\mathcal{O}G$-modules associated with b. We write $L^0(G)$ instead of $L^0(G, 1)$ if $\mathcal{O}G$ is indecomposable as an $\mathcal{O}$-algebra. We denote by C_n a cyclic group of order n. For G a finite group and $\alpha \in H^2(G; k^\times)$ we denote by $k_\alpha G$ the *twisted group algebra* which is equal to kG as a k-vector space, endowed with the bilinear multiplication $x \cdot y = \alpha(x, y)(xy)$, for x, $y \in G$, where α denotes abusively a 2-cocycle representing the class α (and one verifies that this construction is, up to isomorphism, independent of the choice of this 2-cocycle). If b is a block of kG with a defect group P and if c is the corresponding block of $kN_G(P)$ then c is a sum of $N_G(P)$-conjugate block idempotents of $kC_G(P)$; for any choice e of such a block idempotent of $kC_G(P)$ the group $E = N_G(P, e)/PC_G(P)$ is called the *inertial quotient of* b. This is a p'-subgroup of the outer automorphism group of P, unique up to conjugacy by an element in $N_G(P)$, hence lifts uniquely, up to conjugacy, to a p'-subgroup of $\mathrm{Aut}(P)$, still abusively denoted E and called inertial quotient. By there is $\alpha \in H^2(E; k^\times)$ such that $kN_G(P)c$ is Morita equivalent to the twisted group algebra $k_\alpha(P \rtimes E)$, where α is extended trivially from E to $P \rtimes E$. We review some of the standard reduction techniques due to Dade, Fong, Külshammer, Puig, Reynolds. The reduction techniques work irrespective of the characteristic, so for now, p is an arbitrary prime.

Proposition 2.1 (Fong-Reynolds reduction [33], [67])**.** *Let G be a finite group, N a normal subgroup of G, c a G-stable block of kN and b a block of kG such that $bc = b$. Let P be a defect group of b. If $P \cap N = 1$ then kNc is a block of defect zero and we have a canonical isomorphism*

$$kGc \cong kNc \otimes_k k_\alpha(G/N)$$

for some $\alpha \in H^2(G/N; k^\times)$ such that kGb is Morita equivalent to a block $\hat{b}$ of a finite central p'-extension of G/N via a bimodule with diagonal vertex ΔP and endo-permutation source.

See for instance [40, 4.4] for an explicit description of this isomorphism. The previous result has been generalised by Külshammer as follows:

Proposition 2.2 (Külshammer [46, Proposition 5, Theorem 7])**.** *Let G be a finite group, N a normal subgroup of G, c a G-stable block of kN and b a block of kG such that $bc = b$. Suppose that for any $x \in G$ conjugation by x induces an inner automorphism of kNc. Then there is a canonical isomorphism*

$$kGc \cong kNc \ \otimes_{Z(kNc)} \ Z(kNc)_\alpha(G/N)$$

for some $\alpha \in H^2(G/N; (Z(kNc))^\times)$. Moreover, if N contains a defect group P of b then the blocks kGb and kNc are source algebra equivalent.

As before, this isomorphism can be described explicitly; see e.g. [24, 2.1].

Proposition 2.3 (Dade [25, 3.5, 7.7])**.** *Let G be a finite group, N a normal subgroup of G, c a G-stable block of kN and b a block of kG such that $bc = b$. Suppose that no element $x \in G - N$ acts as inner automorphism on kNc. Then $b = c$ and G/N has order prime to p.*

Proposition 2.4 (Puig [66, 4.3])**.** *Let G be a finite group, N a normal subgroup of G, c a G-stable block of kN and b a block of kG such that $bc = b$. Suppose that b is nilpotent. Then the block algebra of c is Morita equivalent to its Brauer correspondent via a Morita equivalence induced by a bimodule with endo-permutation source; in particular, c satisfies Alperin's weight conjecture.*

3. Background material on blocks with defect group $C_2 \times C_2 \times C_2$

We assume from now on that $p = 2$. Let $P \cong C_2 \times C_2 \times C_2$ be an elementary abelian group of order 8. The order of $\mathrm{GL}_3(2)$ is $8 \cdot 21$, from which one easily deduces that a non-trivial subgroup E of $\mathrm{Aut}(P)$ of odd order has either order 3 or 7, or is a Frobenius group of order 21. In all cases, E has a trivial Schur multiplier, and hence any block with a normal defect group P has as source algebra $k(P \rtimes E)$. What is unusual in this case is that the number of characters at the local level does not depend on fusion (this is well-known; we include a sketch of a proof for the convenience of the reader):

Proposition 3.1. *Let P be an elementary abelian group of order 8 and E a subgroup of $\mathrm{Aut}(P)$ of odd order. The group $P \rtimes E$ has 8 ordinary irreducible characters.*

Proof. This is trivial if $E = 1$. Suppose that $|E| = 3$. Then E fixes an involution in P, hence $P \rtimes E \cong C_2 \times (V_4 \rtimes C_3)$, from which the statement follows. If $|E| = 7$ then $P \rtimes E$ is a Frobenius group with E acting transitively on the involutions in P, which implies the result. The only remaining case is where E is a Frobenius group $C_7 \rtimes C_3$. In that case, E has 5 characters, three of degree 1 with C_7 in the kernel, and two more of degree 3 corresponding to the non-trivial C_3-orbits in the character group of C_7. Using the fact that the degrees of irreducible characters of $P \rtimes E$ divide $|E|$ and that the square of their degrees sums up to $|P \rtimes E| = 168$ one finds that there are three further irreducible characters of degree 7. □

Landrock showed the inequality $\ell(b) \leq 8$ of Alperin's weight conjecture for 2-blocks b with an elementary abelian defect group P of order 8 without the classification of finite simple groups. Landrock's results are more precise in that they also include information about defects and heights of characters; we briefly recall these notions. Given a block b of $\mathcal{O}G$ or of kG with a defect group

P of order p^d, the *defect* of a character χ in the set $\mathrm{Irr}_K(G,b)$ of irreducible K-valued characters associated with b is the integer $d(\chi)$ such that $p^{d(\chi)}$ is the largest power of p dividing the rational integer $\frac{|G|}{\chi(1)}$. It is well-known that $d(\chi) \leq d$; the integer $h(\chi) = d - d(\chi)$ is called the *height* of χ. There is always at least one character in $\mathrm{Irr}_K(G,b)$ having height zero, and it has been conjectured by Brauer that P is abelian if and only if all characters in $\mathrm{Irr}_K(G,b)$ have height zero. The following summary of some of Landrock's results in [48] implies in particular that Alperin's weight conjecture holds for blocks with an elementary abelian defect group of order 8 if and only if all characters in those blocks have height zero.

Proposition 3.2 (Landrock, [48, 2.1, 2.2, 2.3])**.** *Let G be a finite group and b a block of kG with an elementary abelian defect group P of order 8 and inertial quotient $E \leq \mathrm{Aut}(P)$. Then $5 \leq |\mathrm{Irr}_K(G,b)| \leq 8$. If $|\mathrm{Irr}_K(G,b)| = 8$ then all characters in $\mathrm{Irr}_K(G,b)$ have height zero. If $|\mathrm{Irr}_K(G,b)| < 8$ then exactly four characters in $\mathrm{Irr}_K(G,b)$ have height zero, the remaining characters have height one, and $\ell(b) = 4$. Moreover, the following hold.*

(i) If E has order 1 then $|\mathrm{Irr}_K(G,b)| = 8$ and $\ell(b) = 1$.

(ii) If E has order 3 then $|\mathrm{Irr}_K(G,b)| = 8$ and $\ell(b) = 3$.

(iii) If E has order 7 then either $|\mathrm{Irr}_K(G,b)| = 5$, $\ell(b) = 4$ or $|\mathrm{Irr}_K(G,b)| = 8$, $\ell(b) = 7$.

(iv) If E has order 21 then either $|\mathrm{Irr}_K(G,b)| = 7$, $\ell(b) = 4$ or $|\mathrm{Irr}_K(G,b)| = 8$, $\ell(b) = 5$.

Besides Landrock's original proof it is also possible to prove this as a consequence of stronger results obtained later: the case $|E| = 1$ is a particular case of nilpotent blocks [17], the case $|E| = 3$ follows from Watanabe [76, Theorem 1]. In the case $|E| = 7$, the group E acts regularly on $P - \{1\}$, and by a result of Puig in [64] there is a *stable equivalence of Morita type* (cf. [15, §5]) between $\mathcal{O}Gb$ and $\mathcal{O}(P \rtimes E)$. Any such stable equivalence induces an isometry $L^0(G,b) \cong L^0(P \rtimes E)$ between the generalised character groups which vanish on p-regular elements; in this case, these groups have rank one and are generated by an element of norm 8, whence the inequality $|\mathrm{Irr}_K(G,b)| \leq 8$. If $|E| = 21$ then there is again a stable equivalence of Morita type, by a result of Rouquier in [71]. Again by calculating a basis of $L^0(P \rtimes E)$ - which in this case has rank 3 with a basis consisting of three elements of norm four, one also gets this inequality. See for instance [45] for an exposition of the well-known technique exploiting partial isometries induced by stable equivalences of Morita type; this will be used in the proof of Theorem 5.1. The following observation is a slight refinement of Proposition 3.2(iii) in the case where the inertial quotient has order 7 and $|\mathrm{Irr}_K(G,b)| < 8$.

Proposition 3.3. *Let G be a finite group, b a block of kG with an elementary abelian defect group P of order 8 and inertial quotient E of order 7. Suppose that b does not satisfy Alperin's weight conjecture. Then there is a labelling $\mathrm{Irr}_K(G,b) = \{\chi_i \mid 1 \leq i \leq 5\}$ with the following properties:*

(i) χ_1 has height one and χ_i has height zero, for $2 \leq i \leq 5$.

(ii) The group $L^0(G,b)$ has rank 1 and a basis element of the form $2\chi_1 - \sum_{i=2}^{5} \delta_i\chi_i$ for some signs $\delta_i \in \{\pm 1\}$, $2 \leq i \leq 5$; moreover, at least one of the δ_i is positive.

(iii) If $i, j \in \{2,3,4,5\}$ such that $\chi_i(1) = \chi_j(1)$ then $\delta_i = \delta_j$.

Proof. Statement (i) is just a reformulation of 3.2 (iii). Using Puig's stable equivalence of Morita type [64, 6.8] we get that $L^0(G,b) \cong L^0(P \rtimes E)$, which is a free abelian group of rank one with a basis element of norm 8. The only way to write a norm 8 element in $L^0(G,b)$ with less than 8 characters is with five characters, exactly one of which shows up with multiplicity 2, and then this character must have height one, as follows from comparing character degrees in conjunction with the fact that every generalised character in $L^0(G,b)$ vanishes at 1. The signs δ_i cannot all be negative because the group $L^0(G,b)$ does not contain any actual non-zero character (again because its elements vanish at 1). This proves (ii). If $i,j \in \{2,3,4,5\}$ then $\delta_i\chi_i - \delta_j\chi_j$ is orthogonal to $L^0(G,b)$, hence a generalised projective character. In particular, at 1, its value is divisible by the order of a Sylow 2-subgroup of G. But if $\chi_i(1) = \chi_j(1)$ and $\delta_i \neq \delta_j$, this value is $\pm 2\chi_i(1)$, which cannot be divisible by the order of a Sylow 2-subgroup of G as χ_i has height zero. The result follows. $\square$

It is not known in general whether a Morita equivalence between two block algebras preserves their local structures, but some easy standard block theoretic arguments show that this is true, even for stable equivalences of Morita type, if one of the two blocks has an elementary abelian defect group of order 8.

Proposition 3.4. *Let G, H be finite groups, b a block of kG with an elementary abelian defect group P of order 8 and c a block of kH with a defect group Q. If there is a stable equivalence of Morita type between kGb and kHc then $Q \cong P$ and the blocks b and c have isomorphic inertial quotients (or equivalently, isomorphic fusion systems).*

Proof. A stable equivalence of Morita type preserves the largest elementary divisors of the Cartan matrices of the blocks, and these are equal to the orders of the defect groups, whence $|Q| = |P|$. A stable equivalence of Morita type preserves also the complexity (cf. [4, 5.3.4]) of modules; since the largest complexity of a module in a block is the rank of a defect group, we get that Q has rank 3, and thus $Q \cong P$. Alternatively, a stable equivalence of Morita type preserves the Krull dimension of the Hochschild cohomology rings, which are also known to be equal to the ranks of the defect groups. (This part of the argument is well-known to remain valid for blocks with arbitrary elementary abelian defect groups, but we do not need this here.) Finally, a stable equivalence of Morita type preserves the rank of $L^0(G,b)$, which is equal to $|\mathrm{Irr}_K(G,b)| - \ell(b)$, or also equal to $\sum_{(u,e_u)} \ell(e_u)$, where (u,e_u) runs over a set of representatives of the conjugacy classes of non-trivial (G,b)-Brauer elements. It happens so that this number determines the structure of the inertial quotient E of b. Indeed, by Proposition 3.2, this number is equal to 7 if and only if $|E| = 1$, equal to 5 if and only if $|E| = 3$, equal to 1 if and only if $|E| = 7$, and equal to 3 if and only if $|E| = 21$, whence the result. $\square$

When dealing with the exceptional groups of type $E_7(q)$ we will need a refinement of the preceding result because the finite group of Lie type $E_7(q)$ is a central extension of the simple group of type $E_7(q)$ by an involution, and so Bonnafé-Rouquier's Jordan decomposition [8, §11, Théorème B'] will have to be applied to blocks with a defect group of order 2^4.

Proposition 3.5. *Let G, H be finite groups, b a block of G with a defect group P and c a block of H with a defect group Q. Suppose that there are central involutions $s \in Z(G)$ and $t \in Z(H)$ such that $P/\langle s\rangle$ is elementary abelian of order 8. Denote by $\bar{b}$ and $\bar{c}$ the images of b and c in $kG/\langle s\rangle$ and $kH/\langle t\rangle$, respectively. Suppose that the block algebras kGb and kHc are Morita equivalent. If $\bar{b}$ does not satisfy Alperin's weight conjecture then neither does $\bar{c}$ and the defect groups of $\bar{c}$ are elementary abelian of order 8.*

Proof. Suppose that $\bar{b}$ does not satisfy Alperin's weight conjecture. Since the defect group $P/\langle s\rangle$ of $\bar{b}$ is elementary abelian of order 8 we have $\ell(\bar{b}) = 4$ by Proposition 3.2. Using that the number of isomorphism classes of simple modules is invariant under central 2-extensions and Morita equivalences we get that $\ell(\bar{c}) = 4$. Note that P and Q have the same order since b and c are Morita equivalent, and hence $\bar{c}$ has a defect group $R = Q/\langle t\rangle$ of order 8. If R is cyclic or abelian of rank 2 or isomorphic to one of the non-abelian groups of order 8 then then $\ell(\bar{c}) \in \{1, 3\}$, a contradiction. So, R is elementary abelian. Now it is immediate from Proposition 3.2 that $\bar{c}$ does not satisfy the weight conjecture. □

Remark 3.6. Experts seem to agree that the Morita equivalences from Bonnafé-Rouquier's Jordan decomposition [8, §10, §11] should preserve the local structure of the blocks, but at present there is no written reference for this fact. The two propositions 3.4 and 3.5 circumvent this issue by adhoc methods.

4. Reduction to quasi-simple groups

The part of Brauer's height zero conjecture predicting that all characters in a block with an abelian defect group have height zero has been reduced to blocks of quasi-simple finite groups in work of Berger and Knörr [5]. We need to make sure that in the reduction we can indeed restrict the problem to checking only defect groups of order at most 8; this is not entirely obvious since in Step 6 of the proof of [5, Theorem] the order of the defect group may possibly go up, an issue which arises also in the alternative proof given by Murai in [58, §6].

Theorem 4.1. *Let G be a finite group and b a block of kG with an elementary abelian defect group P of order 8. Suppose that $|G/Z(G)|$ is minimal such that $|\mathrm{Irr}_K(G, b)| < 8$. Then $Z(G)$ has odd order and $G/Z(G)$ is simple. If moreover we also choose $|Z(G)|$ minimal then G is quasi-simple. In addition, the inertial quotient of b is either cyclic of order 7 or a Frobenius group of order 21.*

Proof. If $|Z(G)|$ is even then $P \cap Z(G)$ is non-trivial, hence contains a subgroup Z of order 2. The image $\bar{b}$ of b in kG/Z is then a block of kG/Z with a Klein four defect group P/Z. Thus, since $\bar{b}$ satisfies Alperin's weight conjecture, so does b, a contradiction to the assumption. Thus $Z(G)$ has odd order. Let (P, e) be a maximal (G, b)-Brauer pair and set $E = N_G(P, e)/C_G(P)$. By Proposition 3.2, the order of E is either 7 or 21. In both cases, E acts transitively on $P - \{1\}$. Thus, if N is a normal subgroup of G then either $N \cap P = \{1\}$ or $P \leq N$. Moreover, the minimality of $|G/Z(G)|$ implies that if N is a normal subgroup of G containing $Z(G)$ then there is a unique block c of kN covered by b; that is, $bc = b$. Let now N be a maximal normal subgroup of G containing $Z(G)$ and let c be the block of kN satisfying $bc = b$. Consider first the case $P \cap N = \{1\}$. In that case, by Proposition 2.1, we have an isomorphism

$$kGc \cong kNc \otimes_k k_\alpha(G/N)$$

for some $\alpha \in H^2(G/N; k^\times)$ such that kGb is Morita equivalent to a block $\hat{b}$ of a finite central $2'$-extension H of the simple group G/N. Consider next the case $P \leq N$. Let $G[c]$ be the subgroup of G consisting of all $x \in G$ such that conjugation by x induces an inner automorphism of kNc. Since N is maximal normal in G we have either $N = G[c]$ or $G[c] = G$. Suppose first that $G[c] = G$. Then, by Proposition 2.2, we have an isomorphism

$$kGc \cong kNc \ \otimes_{Z(kNc)} \ Z(kNc)_\alpha(G/N)$$

for some $\alpha \in H^2(G/N; (Z(kNc))^\times)$, and the blocks kGb and kNc are source algebra equivalent - but this contradicts the minimality of $|G|$ since source algebra equivalent blocks have in particular the same number of ordinary irreducible characters. Thus we have $N = G[c]$. Then, by Proposition 2.3, we have $b = c$ and G/N has odd order. Since also G/N is simple, this implies that G/N is cyclic of odd prime order ℓ, by Feit-Thompson's Odd Order theorem. By standard results in Clifford theory, any $\eta \in \mathrm{Irr}_K(N,c)$ is either G/N-stable, in which case it extends to exactly ℓ different characters in $\mathrm{Irr}_K(G,b)$, or $\mathrm{Ind}_N^G(\eta) \in \mathrm{Irr}_K(G,b)$. It follows that

$$|\mathrm{Irr}_K(G,b)| = \ell \cdot m + r < 8$$

where m is the number of characters in $\mathrm{Irr}_K(N,c)$ fixed by G/N and where r is the number of non-trivial G/N-orbits in $\mathrm{Irr}_K(N,c)$. Since $\ell \geq 3$ we have $m \leq 2$. Using induction, we have

$$8 = |\mathrm{Irr}_K(N,c)| = \ell \cdot r + m$$

In all possible choices of ℓ, m, r satisfying this equality we get the contradiction $\ell \cdot m + r \geq 8$. Thus the assumption $P \leq N$ is not possible, and therefore the above implies that $G/Z(G)$ is simple. Finally, since $G/Z(G)$ is simple we have $G = Z(G)[G,G]$, so $G/[G,G]$ acts trivially on all characters of $[G,G]$, and so $|\mathrm{Irr}_K(G,b)| = |\mathrm{Irr}_K([G,G],d)|$, where d is a block of $[G,G]$ satisfying $bd = b$. After repeating this, if necessary, we also may assume that G is perfect, hence quasi-simple. □

5. Perfect isometries

Using Rouquier's stable equivalence for blocks with an elementary abelian defect group of order 8, described in the Appendix below, we show that Alperin's weight conjecture implies the character theoretic version of Broué's abelian defect conjecture for these blocks:

Theorem 5.1. *Let G be a finite group and let b be a block of $\mathcal{O}G$ with an elementary abelian defect group P of order 8. Set $H = N_G(P)$ and denote by c the block of $\mathcal{O}H$ with defect group P corresponding to b via the Brauer correspondence. Suppose that K is large enough for b and c. If $|\mathrm{Irr}_K(G,b)| = |\mathrm{Irr}_K(H,c)|$ then the blocks b and c are isotypic; in particular, there is a perfect isometry $\mathbb{Z}\mathrm{Irr}_K(G,b) \cong \mathbb{Z}\mathrm{Irr}_K(H,c)$. In particular we have $Z(\mathcal{O}Gb) \cong Z(\mathcal{O}Hc)$.*

See [13, 6.1, 6.2], [14], [15] for more precise versions of Broué's abelian defect conjecture, as well as background material on perfect isometries and isotypies.

Proof of Theorem 5.1. We refer to [45, §§2, 3] for notation and an expository account of the standard techniques on extending partial isometries induced by stable equivalences of Morita type. By a result of Rouquier in [71], there is a stable equivalence of Morita type between the block algebras of b and of c over $\mathcal{O}$ (a proof of this result is given in Theorem 21.1 below) given by a bounded complex of bimodules whose indecomposable summands all have diagonal vertices and trivial source. Denote by E the inertial quotient of b. Since the block algebra $\mathcal{O}Hc$ is Morita equivalent to $\mathcal{O}(P \rtimes E)$ via a bimodule with diagonal vertex and trivial source this implies that there is a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}(P \rtimes E)$, induced by a bounded complex of bimodules with diagonal vertices and trivial source. It is well-known (see e.g. [45, 3.1]) that any such stable equivalence induces an isometry $L^0(P \rtimes E) \cong L^0(G,b)$. It suffices to show that this partial isometry extends to an isometry $\mathbb{Z}\mathrm{Irr}_K(P \rtimes E) \cong \mathbb{Z}\mathrm{Irr}_K(G,b)$ because any such extension is then a p-permutation equivalence by [45, 3.3], hence induces an isotypy by [50, Theorem 1.4]. We do this by running through all possible inertial quotients E.

If $E = \{1\}$ the block b is nilpotent, hence Morita equivalent to $\mathcal{O}P$, and so the result holds trivially in this case. Assume that $|E| = 3$. Then $P \rtimes E \cong C_2 \times A_4$, and hence we can list the eight ordinary irreducible characters of $P \rtimes E$ in such a way that the three characters of the projective indecomposable $\mathcal{O}(P \rtimes E)$-modules are of the form $\chi_i + \chi_{i+3} + \chi_7 + \chi_8$ where $1 \leq i \leq 3$. Thus a basis of $L^0(P \rtimes E)$ is of the form

$$\{\chi_1 - \chi_4, \chi_2 - \chi_5, \chi_3 - \chi_6, \chi_7 - \chi_8, \chi_1 + \chi_2 + \chi_3 - \chi_7\}$$

The four elements of norm 2 in this basis must be sent to norm 2 elements under the isometry $L^0(P \rtimes E) \cong L^0(G,b)$ no two of which involve a common irreducible character in $\mathrm{Irr}_K(G,b)$, and hence are mapped to elements of the form $\delta_1(\eta_1 - \eta_4)$, $\delta_2(\eta_2 - \eta_5)$, $\delta_3(\eta_3 - \eta_6)$, $\delta_7(\eta_7 - \eta_8)$, for some labelling $\mathrm{Irr}_K(G,b) = \{\eta_i \mid 1 \le i \le 8\}$ and some signs δ_i. We may then choose notation (after possibly exchanging η_1 and η_4 etc.) in such a way that the image in $L^0(G,b)$ of the norm four element $\chi_1+\chi_2+\chi_3-\chi_7$ is equal to $\delta_1\eta_1+\delta_2\eta_2+\delta_3\eta_3-\delta_7\eta_7$. Setting $\delta_{i+3} = \delta_i$ for $1 \le i \le 3$, and $\delta_8 = \delta_7$ it follows that the map sending χ_i to $\delta_i\eta_i$ induces an isometry $\mathbb{Z}\mathrm{Irr}_K(P \rtimes E) \cong \mathbb{Z}\mathrm{Irr}_K(G,b)$ extending the isometry $L^0(P \rtimes E) \cong L^0(G,b)$ as required.

Assume next that $|E| = 7$. Then $P \rtimes E$ is a Frobenius group, whose seven characters of the projective indecomposable modules are of the form $\chi_i + \chi_8$ with $1 \le i \le 7$, for some labelling $\mathrm{Irr}_K(P \rtimes E) = \{\chi_i \mid 1 \le i \le 8\}$; the characters χ_i, $1 \le i \le 7$ have P in their kernel, and χ_8 is induced from a nontrivial character of P to $P \rtimes E$. The group $L^0(P \rtimes E)$ has rank 1, with a basis element $\sum_{i=1}^7 \chi_i - \chi_8$. This element has norm 8, hence its image in $L^0(G,b)$ has norm 8 as well. Moreover, all irreducible characters in $\mathrm{Irr}_K(G,b)$ have to be involved in this element. Since we assume that Alperin's weight conjecture holds for b, we have $|\mathrm{Irr}_K(G,b)| = 8$, and so the image of this element in $L^0(G,b)$ is of the form $\sum_{i=1}^7 \delta_i\eta_i - \delta_8\eta_8$ for some labelling $\mathrm{Irr}_K(G,b) = \{\eta_i \mid 1 \le i \le 8\}$ and some signs δ_i. Again, the map sending χ_i to $\delta_i\eta_i$ induces the required isometry $\mathbb{Z}\mathrm{Irr}_K(P \rtimes E) \cong \mathbb{Z}\mathrm{Irr}_K(G,b)$.

Finally, assume that $|E| = 21$. Then E is itself a Frobenius group, isomorphic to $C_7 \rtimes C_3$ with the obvious nontrivial action of C_3 on C_7. The group E has 5 ordinary irreducible characters, hence $\mathcal{O}(P \rtimes E)$ has five isomorphism classes of projective indecomposable modules, and thus $L^0(G,b)$ has rank 3. We can label $\mathrm{Irr}_K(P \rtimes E) = \{\chi_i \mid 1 \le i \le 8\}$ in such a way that χ_1, χ_2, χ_3 have degree 1, the characters χ_4, χ_5 have degree 3 and χ_6, χ_7, χ_8 have degree 7. An easy calculation shows that $L^0(P \rtimes E)$ has a basis of the form

$$\{\chi_6 - \chi_4 - \chi_5 - \chi_1,\ \chi_7 - \chi_4 - \chi_5 - \chi_2,\ \chi_8 - \chi_4 - \chi_5 - \chi_3\}$$

consisting of three elements of norm 4, such that any two different of these basis elements involve two common irreducible characters. Thus the same is true for $L^0(G,b)$. Using again the hypothesis $|\mathrm{Irr}_K(G,b)| = 8$ one deduces that $L^0(G,b)$ has a basis of the form

$$\{\delta_6\eta_6 - \delta_4\eta_4 - \delta_5\eta_5 - \delta_1\eta_1,\ \delta_7\eta_7 - \delta_4\eta_4 - \delta_5\eta_5 - \delta_2\eta_2,\ \delta_8\eta_8 - \delta_4\eta_4 - \delta_5\eta_5 - \delta_3\eta_3\}$$

for some labelling $\mathrm{Irr}_K(G,b) = \{\eta_i \mid 1 \le i \le 8\}$ and some signs δ_i. As before, the map sending χ_i to $\delta_i\eta_i$ induces the required isometry $\mathbb{Z}\mathrm{Irr}_K(P \rtimes E) \cong \mathbb{Z}\mathrm{Irr}_K(G,b)$. Since a perfect isometry induces an isomorphism between centers, the result follows. $\square$

6. Sporadic finite simple groups

Let G be a finite group and b a block of $\mathcal{O}G$. The *kernel* $\mathrm{Ker}_G(b)$ of b is defined by

$$\mathrm{Ker}_G(b) = \bigcap_{\chi \in \mathrm{Irr}_K(G,b)} \mathrm{Ker}(\chi),$$

see [10, §3]. By [10, Proposition (3B)] we have $\mathrm{Ker}_G(b) = O_{p'}(G) \cap \mathrm{Ker}(\chi)$ for any $\chi \in \mathrm{Irr}_K(G,b)$. Hence, $\mathrm{Ker}_G(b)$ is a normal p'-subgroup of G, See [59, Chap.5, Theorem 8.1] for an exposition of this material. We say that b is *faithful* if $\mathrm{Ker}_G(b) = 1$.

Proposition 6.1. *Let G be a quasi-simple finite group such that p divides $|G|$.*
(i) We have $O_{p'}(Z(G)) = O_{p'}(G)$ and $\mathrm{Ker}_G(b) = O_{p'}(Z(G)) \cap \mathrm{Ker}(\chi)$ for any $\chi \in \mathrm{Irr}_K(G,b)$.

(ii) Set $\bar{G} = G/\mathrm{Ker}_G(b)$ and denote by $\bar{b}$ the image of b in $\mathcal{O}\bar{G}$. Then $\bar{b}$ is a faithful block of $\mathcal{O}\bar{G}$ and the canonical map $G \to \bar{G}$ induces an $\mathcal{O}$-algebra isomorphism $\mathcal{O}Gb \cong \mathcal{O}\bar{G}\bar{b}$.

Proof. Statement (i) follows from Brauer's result mentioned above, and (ii) is an easy consequence of [59, Chap.5, Theorem 8.8]. □

The following table is due to Noeske [60]. By Proposition 6.1(ii) it is enough to consider faithful blocks.

Proposition 6.2 ([60])**.** *The following is a list of all faithful non-principal 2-blocks with non-cyclic abelian defect groups of sporadic simple groups and their covers. Each number in the 2nd column corresponds to the number attached to each block in the Modular Atlas [56].*

group	blocks b	defect groups	$k(b)$	$\ell(b)$
M_{12}	2	$C_2 \times C_2$	4	3
$12.M_{22}$	4, 5	$C_2 \times C_2$, $C_2 \times C_2$	4, 4	1, 1
J_2	2	$C_2 \times C_2$	4	3
HS	2	$C_2 \times C_2$	4	3
Ru	2	$C_2 \times C_2$	4	3
Co_3	2	$C_2 \times C_2 \times C_2$	8	5
$2.Fi_{22}$	3	$C_2 \times C_2$	4	1
$Fi_{24}{}'$	2	$C_2 \times C_2$	4	3

Proposition 6.3. *Let G be a quasi-simple finite group such that $G/Z(G)$ is a sporadic simple group, and let b be a block of kG with an elementary abelian defect group of order 2^r for some integer $r \geq 3$. Then $r = 3$ and either b is the principal block of kJ_1 or a non-principal block of kCo_3. In both cases we have $|\mathrm{Irr}_K(G, b)| = 8$; in particular, Alperin's weight conjecture holds for b.*

Proof. If b is a principal block then $r = 3$ and $G = J_1$, hence the result follows from [48, Theorem 3.8]. Suppose that b is a non-principal block; by Proposition 6.1 we may assume that b is faithful. Proposition 6.2 implies that $r = 3$, $G = Co_3$ and $|\mathrm{Irr}_K(G, b)| = 8$. □

7. Finite simple groups of Lie type with exceptional Schur multipliers

The Schur multipliers of finite groups of Lie type tend to be 'generic' (that is, dependent only on the series to which the group belongs) except in a few cases of low rank where they are larger; see [39, Definition 6.1.3]. We consider in this section the groups of Lie type from [39, Table 6.1.3] defined over a field of odd characteristic.

Proposition 7.1 ([39, Table 6.1.3, p.313], [56])**.** *The finite simple group G of Lie type defined over a field of odd characteristic with exceptional Schur multipliers are as follows:*

G	$A_1(9) \cong A_6$	${}^2A_3(3) \cong \mathrm{PSU}_4(3)$	$B_3(3) \cong P\Omega_7(3)$	$G_2(3)$
$\mathcal{M}e$	3	3, 3	3	3
$M(G)$	6	12	6	3

where $\mathcal{M}e$ denotes the elementary divisors of the exceptional parts of the Schur multipliers $M(G)$ of G.

Proposition 7.2. *If G is a quasi-simple finite group such that $Z(G)$ has odd order and $G/Z(G)$ is of Lie type either $A_1(9)$, ${}^2A_3(3)$, $B_3(3)$ or $G_2(3)$, then G has no 2-blocks b with elementary abelian defect group of order 2^r, where $r \geq 3$.*

Proof. For the isomorphisms $A_1(9) \cong A_6$ and ${}^2A_3(3) \cong \mathrm{PSU}_4(3)$ and $B_3(3) \cong P\Omega_7(3)$ (also denoted $O_7(3)$ in the Atlas [23, p. 106]), used already in the previous Proposition, see [38, p.8, Table I]. If $G/Z(G) \cong A_6$ then G is isomorphic to A_6 or $3.A_6$; in both cases G has no 2-blocks with an elementary abelian defect group of order 2^r, $r \geq 3$ by [56], or by 8.2 below. If $G/Z(G)$ is isomorphic to one of $\mathrm{PSU}_4(3)$, $P\Omega_7(3)$, $G_2(3)$ then again G has no 2-blocks with elementary abelian defect group of order 2^r, $r \geq 3$, by [56]. □

8. Alternating groups

We denote in this section by A_n the alternating group of degree n, where n is a positive integer.

Proposition 8.1. *If $G \cong A_n$ for some $n \geq 5$ then G has no 2-blocks with an elementary abelian defect group of order 2^r, where $r \geq 3$.*

Proof. By [44, 1.2, 1.3, 1.4, 1.7], a 2-block of an alternating group A_n with defect group P is source algebra equivalent to a block of an alternating group A_m for some $m \leq n$ having P as Sylow 2-subgroup. But there is no alternating group with an elementary abelian Sylow 2-subgroup of order 2^r, $r \geq 3$. □

Proposition 8.2. *If G is $3.A_6$ or $3.A_7$, then G has no 2-block with an elementary abelian defect group of order 2^r, where $r \geq 3$.*

Proof. This is clear since Sylow 2-subgroups of G are dihedral of order 8, see [23, p.4, p.10]. □

9. Finite groups of Lie type in characteristic 2

Proposition 9.1. *Let G be a quasi-simple group such that $G/Z(G)$ is a finite group of Lie type in characteristic 2. Suppose that $Z(G)$ has odd order. Let b be a block of G having an elementary abelian defect group of order 2^r for some integer $r \geq 3$. Then $G \cong \mathrm{PSL}_2(2^r)$, the block b is the principal block of G, and Alperin's weight conjecture holds for b.*

Proof. Consider first the case where $G/Z(G)$ is not isomorphic to the Tits simple group ${}^2F_4(2)'$, $\mathrm{PSp}_4(2)' \cong A_6$, or $G_2(2)' \cong \mathrm{PSU}_3(3)$. Then, by [24, Proposition 8.7], the defect groups of any 2-block of G are either trivial or the Sylow 2-subgroups of G. It is well-known that the only finite groups of Lie type in characteristic 2 having abelian Sylow 2-subgroups are the groups $\mathrm{PSL}_2(2^r)$, and hence $G \cong \mathrm{PSL}_2(2^r)$, where we use that the Schur multiplier of $\mathrm{PSL}_2(2^r)$ is trivial. By [13, A 1.3], the principal block is isotypic to its Brauer correspondent; in particular, Alperin's weight conjecture holds. The group ${}^2F_4(2)'$ has trivial Schur multiplier and by the pages on decomposition numbers in the Modular Atlas [56], ${}^2F_4(2)'$ has three 2-blocks: the principal block (of defect 11) and two defect zero blocks; ${}^2F_4(2)'$ has no block with an elementary abelian defect group of order 2^r, $r \geq 3$. The case $\mathrm{PSp}_4(2) \cong A_6$ is already checked in §7 and in Proposition 8.2. Finally, $\mathrm{PSU}_3(3)$ has trivial Schur multiplier and again by the pages on decomposition numbers in the Modular Atlas [56], $\mathrm{PSU}_3(3)$ has besides the principal block (of defect 5) two blocks of defect zero; in particular, $\mathrm{PSU}_3(3)$ has no blocks with an elementary abelian defect group of order 2^r, $r \geq 3$. □

Remark 9.2. For the purpose of the proof of 1.1 one could have excluded the Tits simple group ${}^2F_4(2)'$ and A_6 also by observing that its order is not divisible by 7, and hence the inertial quotient of a hypothetical block with elementary abelian defect group of order 8 can only be trivial or cyclic of order 3, in which case Alperin's weight conjecture holds by Proposition 3.2 above.

10. Further background results on finite reductive groups

The book [26], especially Chapters 13 and 14 is a useful reference for the first part of this section. The following notation will be in effect for this section. Let r and ℓ be disctinct primes and let q be a power of r. Let $\mathbf{G}$ a connected reductive group over $\bar{\mathbb{F}}_q$ and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $(\mathbf{G}^*, F^*)$ be a pair in duality with $(\mathbf{G}, F)$ with respect to some choice of an F-stable (respectively F^*-stable) maximal torus of $\mathbf{G}$ (respectively $\mathbf{G}^*$) and with respect to a fixed isomorphism $\bar{\mathbb{F}}_q^\times \cong (\mathbb{Q}/\mathbb{Z})_{r'}$ and a fixed embedding $\bar{\mathbb{F}}_q^\times \hookrightarrow \bar{\mathbb{Q}}_\ell^\times$. For an F^*-stable semi-simple element s of $\mathbf{G}^*$, we denote by $\mathcal{E}(\mathbf{G}^F, (s)) \subseteq \mathrm{Irr}_{\bar{\mathbb{Q}}_\ell}(G)$ the subset of characters corresponding to the geometric conjugacy class (s) and by $\mathcal{E}(\mathbf{G}^F, [s])$ the subset of characters corresponding to the rational conjugacy class $[s]$; the geometric (and rational) class of the trivial element will be just denoted $\mathcal{E}(\mathbf{G}^F, 1)$. The elements of $\mathcal{E}(\mathbf{G}^F, 1)$ are called the unipotent characters of $\mathbf{G}^F$. We set $\mathbf{C}(s) = C_{\mathbf{G}^*}(s)$, $\mathbf{C}^\circ(s) = C_{\mathbf{G}^*}(s)^\circ$, the connected component of $\mathbf{C}(s)$, $\bar{\mathbf{C}}^\circ(s) = \mathbf{C}^\circ(s)/Z(\mathbf{C}^\circ(s))$, the quotient of the connected centraliser by its centre, and $\mathbf{C}^\circ(s)' = [\mathbf{C}^\circ(s), \mathbf{C}^\circ(s)]$ the derived subgroup of the connected centraliser. Set $a_s = \frac{|\mathbf{C}(s)^{F^*}|}{|\mathbf{C}^\circ(s)^{F^*}|}$, $\mathbf{Z}^\circ(s) = Z(\mathbf{C}^\circ(s))$ and $z(s) = |\mathbf{Z}^\circ(s)^{F^*}|$. For any positive integer m we denote by m_+ the highest power of 2 dividing m.

10.1. **Jordan decomposition of characters.** By the work of Lusztig ([51, Theorem 4.23], [52, Proposition 5.1], see also [26, 13.24]), there is a bijection between $\mathcal{E}(\mathbf{G}^F, [s])$ and the set $\mathcal{E}(\mathbf{C}(s)^{F^*}, 1)$ of unipotent characters of $\mathbf{C}(s)^{F^*}$ such that if $\chi \in \mathcal{E}(\mathbf{G}^F, [s])$ corresponds to $\tau \in \mathcal{E}(\mathbf{C}(s)^{F^*}, 1)$, then

$$\chi(1) = \frac{|\mathbf{G}^F|_{r'}}{|\mathbf{C}(s)^{F^*}|_{r'}} \tau(1). \tag{1}$$

Here we note that if $\mathbf{C}(s)$ is not connected, then $\mathcal{E}(\mathbf{C}(s)^{F^*}, 1)$ is defined to be the set of irreducible characters of $\mathbf{C}(s)^{F^*}$ covering the set $\mathcal{E}(\mathbf{C}^\circ(s)^{F^*}, 1)$ of unipotent characters of $\mathbf{C}^\circ(s)^{F^*}$. By standard Clifford theory, if τ is an irreducible character of $\mathbf{C}(s)^{F^*}$ covering an irreducible character λ of $\mathbf{C}^\circ(s)^{F^*}$, then $\tau(1) = a\lambda(1)$, for an integer a dividing $a(s)$. Thus, to each element of $\mathcal{E}(\mathbf{G}^F, (s))$ is associated a $\mathbf{C}(s)^{F^*}$-orbit of $\mathcal{E}(\mathbf{C}^\circ(s)^{F^*}, 1)$ such that if $\chi \in \mathcal{E}(\mathbf{G}, (s))$ corresponds to the orbit of $\lambda \in \mathcal{E}(\mathbf{C}^\circ(s)^{F^*}, 1)$, then

$$\chi(1) = \frac{|\mathbf{G}^F|_{r'}}{a_\chi |\mathbf{C}^\circ(s)^{F^*}|_{r'}} \lambda(1) \tag{2}$$

for some integer a_χ dividing a_s.

Restriction induces a degree preserving bijection $\lambda \mapsto \lambda'$ between the sets $\mathcal{E}(\mathbf{C}^\circ(s)^{F^*}, 1)$ and $\mathcal{E}(\mathbf{C}^\circ(s)'^{F^*}, 1)$ and there is also a degree preserving bijection $\lambda \to \bar{\lambda}$ between $\mathcal{E}(\mathbf{C}^\circ(s)^{F^*}, 1)$ and $\mathcal{E}(\bar{\mathbf{C}}^\circ(s)^{F^*}, 1)$ (cf. [19, Proposition 3.1]). Further, the group $\bar{\mathbf{C}}^\circ(s)^{F^*} = \prod_\omega \bar{\mathbf{C}}^{\circ F^*}_\omega$, where ω runs through the F-orbits of the Dynkin diagram Δ_s of $\bar{\mathbf{C}}^\circ(s)$, and for each ω, $\bar{\mathbf{C}}^\circ_\omega(s)$ is the direct product of subgroups of $\bar{\mathbf{C}}^\circ(s)$ corresponding to the elements of ω. The elements of $\mathcal{E}(\bar{\mathbf{C}}^\circ(s)^{F^*}, 1)$ are products $\prod_\omega \phi_\omega$ where ϕ_ω is a unipotent character of $\bar{\mathbf{C}}^\circ(s)^{F^*}_\omega$ for each ω. Tracing through the above bijections, and noting that

$$|\mathbf{C}^\circ(s)^{F^*}| = z(s)|\mathbf{C}^\circ(s)'^{F^*}| = z(s)|\bar{\mathbf{C}}^\circ(s)^{F^*}|$$

it follows that if $\chi \in \mathcal{E}(\mathbf{G},(s))$ corresponds to the orbit of $\lambda \in \mathcal{E}(\mathbf{C}^\circ(s)^{F^*},1)$, and λ corresponds to to the unipotent character λ' of $\mathbf{C}^\circ(s)'^{F^*}$ and to the unipotent character $\bar\lambda := \prod_\omega \phi_\omega$ of $\bar{\mathbf{C}}^\circ(s)^{F^*}$, then

$$\begin{aligned}\chi(1) &= \frac{|\mathbf{G}^F|_{r'}}{z(s)a_\chi|\mathbf{C}^\circ(s)'^{F^*}|_{r'}}\lambda'(1)\\ &= \frac{|\mathbf{G}^F|_{r'}}{z(s)a_\chi}\prod_\omega \frac{\phi_\omega(1)}{|\bar{\mathbf{C}}^{\circ F^*}_\omega(s)|_{r'}}. \end{aligned}\tag{3}$$

The above correspondences have the following consequence, which we record for use in later sections. If r is odd, then

$$\text{2-defect of } \chi = \alpha_\chi + \zeta_s + \text{2-defect of } \lambda' \tag{4}$$

$$= \alpha_\chi + \zeta_s + \sum_\omega (\text{2-defect of } \phi_\omega) \tag{5}$$

where $2^{\zeta_s} = z(s)_+$ and $2^{\alpha_\chi} = |a_\chi|_+$. We note that we get an analogous formula for ℓ-defects, for any prime ℓ different from r.

10.2. **Jordan decomposition of blocks.** As is the case of many sources cited below, we divert in this section from our previous notation and use the prime ℓ, instead of p, for the characteristic of k, which is assumed to be different from the defining characteristic r. We identify without further comment the sets $\mathrm{Irr}_{\bar{\mathbb{Q}}_\ell}(\mathbf{G}^F)$ and $\mathrm{Irr}_K(\mathbf{G}^F)$. Let t be a semi-simple element of $\mathbf{G}^{*F^*}$ of order prime to ℓ and let $\mathcal{E}_\ell(\mathbf{G}^F,[t]) = \cup_u \mathcal{E}(\mathbf{G}^F,[tu])$, where u runs over the ℓ-elements of $C^\circ_{\mathbf{G}^*}(t)^{F^*}$. By [16, Théorème 2.2], $\mathcal{E}_\ell(\mathbf{G}^F,[t]) = \cup_u \mathcal{E}(\mathbf{G}^F,[tu])$, is a union of ℓ-blocks of $\mathbf{G}$; a block b of $\mathbf{G}^F$ is in this union if and only if $\mathcal{E}(\mathbf{G},[t]) \cap \mathrm{Irr}_K(\mathbf{G}^F,b) \neq \emptyset$ (cf. [41, Theorem 3.1]). In this case, we say that b is in the series $[t]$ or that $[t]$ is the semi-simple label of b. Blocks with semi-simple label $[1]$ are called *unipotent*. Let $\mathbf{L}^*(t)$ be the (necessarily F^*-stable) minimal standard Levi subgroup of $\mathbf{G}^*$-containing $\mathbf{C}(t)$ (if $\mathbf{C}(t)$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$, we take for $\mathbf{L}^*(t)$ the group $\mathbf{G}$ itself) and let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*(t)$. Let $e_t^{\mathbf{G}^F}$ and $e_t^{\mathbf{L}^F}$ be the sum of block idempotents of $\mathcal{O}\mathbf{G}^F$ and $\mathcal{O}\mathbf{L}^F$ in the series $[t]$. Then by Theorem B' of [8], the algebras $\mathcal{O}G^F e_t^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_t^{\mathbf{L}^F}$ are Morita equivalent. Further, if $\mathbf{C}(t)$ is itself a Levi subgroup of $\mathbf{G}^*$, ie. if $\mathbf{C}(t) = \mathbf{L}^*(t)$, then by Theorem 11.8 of [8], $\mathcal{O}\mathbf{G}^F e_t^{\mathbf{G}^F}$ is Morita equivalent to the sum of unipotent blocks $\mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$ of $\mathcal{O}\mathbf{L}^F$. We record a consequence of these results for classical groups for $\ell = 2$.

Theorem 10.1. *Suppose that $\ell = 2$ and either $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ or that $\mathbf{G}$ is simple of classical type B, C or D. Let t be an odd order semi-simple element of $\mathbf{G}^{*F^*}$. Then, $\mathbf{C}(t)$ is a Levi subgroup of $\mathbf{G}^*$. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{C}(t)$ as above. Then $e_t^{\mathbf{G}^F}$ is a block of $\mathcal{O}\mathbf{G}^F$, and $e_1^{\mathbf{L}^F}$ is the principal block of $\mathcal{O}\mathbf{L}^F$. The block algebras $\mathcal{O}\mathbf{G}^F e_t^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$ are Morita equivalent and a Sylow 2-subgroup of $\mathbf{L}^F$ is a defect group of $\mathcal{O}\mathbf{G}^F e_t^{\mathbf{G}^F}$.*

Proof. The element t has odd order, whereas $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ is a 2-group, hence $\mathbf{C}(t)$ is connected. The prime 2 is the only bad prime for $\mathbf{G}$ (if $\mathbf{G}$ is a general linear group, then all primes are good for $\mathbf{G}$). Thus, the order of t is not divisble by any bad prime, which means that $\mathbf{C}^\circ(t)$ is a Levi subgroup of $\mathbf{G}^*$. Thus by the Bonnafé-Rouquier theorem, $\mathcal{O}\mathbf{G}^F e_t^{\mathbf{G}^F}$ and $\mathcal{O}\mathbf{L}^F e_1^{\mathbf{L}^F}$ are

Morita equivalent. Now the components of $\mathbf{L}$ are all of classical types A, B, C or D. Hence by [18, Theorem 13], the principal block of $\mathcal{O}\mathbf{L}^F$ is the unique unipotent 2-block of $\mathcal{O}\mathbf{L}^F$, and the Morita equivalence implies that $e_t^{\mathbf{G}^F}$ is a block of $\mathcal{O}\mathbf{G}^F$. The assertion on defect groups is in [31, Proposition 1.5(ii), (iii)]. □

Corollary 10.2. *Suppose that $\ell = 2$ and either $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ or that $\mathbf{G}$ is simple of classical type B, C or D. Let b be a block of $\mathcal{O}\mathbf{G}^F$. If b has abelian defect groups, then $\mathcal{O}\mathbf{G}^F b$ is nilpotent.*

Proof. Let t be the semi-simple label of b and suppose that b has abelian defect groups. By the theorem, $b = e_t^{\mathbf{G}^F}$ and the Sylow 2-subgroups of $\mathbf{L}^F$ are abelian. Suppose that $[\mathbf{L}, \mathbf{L}] \neq 1$. If $\mathbf{L}$ has a component of type different from A_1, then $[\mathbf{L}, \mathbf{L}]^F$ contains a subquotient isomorphic to $\mathrm{SL}_2(q')$ for some power q' of q (see Theorem 3.2.8 of [39]). If all components of $\mathbf{L}$ are of type A_1, then $[\mathbf{L}, \mathbf{L}]^F$ is a commuting product of finite special linear and projective general linear groups of degree 2. But the Sylow 2-subgroups of $\mathrm{SL}_2(q')$ are quaternion and those of $\mathrm{PGL}_2(q')$ are dihedral of order at least 8 for any odd prime power q' (see 11.1 below), a contradiction. Hence, $\mathbf{L}$ and therefore $\mathbf{L}^F$ is an abelian group. In particular, any block of $\mathcal{O}\mathbf{L}^F$ is nilpotent. Since nilpotent blocks with abelian defect groups are precisely the blocks with a symmetric centre (cf. [61, Theorems 3 and 5] and [57]), any block Morita equivalent to a block of $\mathcal{O}\mathbf{L}^F$ is nilpotent with an abelian defect group, whence the result. Alternatively, Morita equivalences of blocks preserve nilpotence by [65, Theorem 8.2]. □

11. On the 2-local structure of finite classical groups

Let n be a natural number, q an odd prime power and let L denote one of the groups $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{O}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{O}^+_{2n}(q)$, or $\mathrm{O}^-_{2n}(q)$. Let Z be a central subgroup of L contained in $[L, L]$ and set $G = [L, L]/Z$. We gather together a few well-known facts on the Sylow 2-structure of the groups L and G.

Lemma 11.1. *With the notation above,*

(i) If $n \geq 3$, then the Sylow 2-subgroups of G (and hence of $[L, L]$ and L) are non-abelian.

(ii) If $n = 2$, the Sylow 2-subgroups of $[L, L]$ are non-abelian. Further, if $n = 2$ and L is not one of $\mathrm{GL}_2(q)$, $q \equiv \pm 3 \pmod 8$, $\mathrm{GU}_2(q)$, $q \equiv \pm 3 \pmod 8$, or $\mathrm{O}^+_4(q)$, $q \equiv \pm 3 \pmod 8$, then the Sylow 2-subgroups of G are non-abelian.

(iii) If L is one of $\mathrm{GL}_2(q)$, $\mathrm{GU}_2(q)$ or $\mathrm{Sp}_2(q)$ then the Sylow 2-subgroups of $[L, L]$ are generalised quaternion groups. They have order at least 16 if $q \equiv \pm 1 \pmod 8$ and in this case the Sylow 2-subgroups of G are non-abelian. If $q \equiv \pm 3 \pmod 8$, then the Sylow 2-subgroups of $[L, L]$ have order 8 and the Sylow 2-subgroups of G are Klein 4-groups. The Sylow 2-subgroups of $\mathrm{PGL}_2(q)$ are dihedral of order at least 8.

(iv) If L is one of $\mathrm{O}^+_2(q)$, respectively $\mathrm{O}^-_2(q)$, then L is a dihedral group of order $2(q-1)$, respectively $2(q+1)$.

(v) If $L = \mathrm{O}_3(q)$ and if $q \equiv 1 \pmod 4$ then the Sylow 2-subgroups of L are isomorphic to the direct product of a cyclic group of order 2 with a Sylow 2-subgroup of $\mathrm{O}^+_2(q)$ and the Sylow 2-subgroups of $\mathrm{SO}_3(q)$ are isomorphic to the Sylow 2-subgroups of $\mathrm{O}^+_2(q)$. If $L = \mathrm{O}_3(q)$ and $q \equiv 3 \pmod 4$ then the Sylow 2-subgroups of L are isomorphic to the direct product of a cyclic group of order 2 with a Sylow 2-subgroup of $\mathrm{O}^-_2(q)$ and the Sylow 2-subgroups of $\mathrm{SO}_3(q)$ are isomorphic to the Sylow 2-subgroups of $\mathrm{O}^-_2(q)$.

Proof. Statements (iii), (iv), (v) can be found in [21]. The only simple groups with abelian Sylow 2-subgroups are $\mathrm{PSL}_2(q')$, $q' \cong \pm 3 \pmod 8$, $\mathrm{PSL}_2(2^a)$, ${}^2G_2(q')$, $q' = 3^{2u+1}$, $u \geq 1$, ${}^2G_2(3)' \cong \mathrm{PSL}_2(8)$ or J_1 (cf. [74], [3]). Also, if $n \geq 2$, then unless L is one of $\mathrm{O}_4^+(q)$, $\mathrm{GL}_2(2)$, $\mathrm{GL}_2(3)$ or $\mathrm{GU}_2(2)$, the groups G are all quasi-simple. Statement (i) and the second assertion of (ii) are immediate from this. If $L = \mathrm{GL}_2(q)$, or $L = \mathrm{GU}_2(q)$, the second assertion of (ii) is immediate from (iii). Now consider $L = \mathrm{O}_4^+(q)$. Then L contains a subgroup isomorphic to $\mathrm{GL}_2(q)$ or to $\mathrm{GU}_2(q)$ (as a centraliser of a semi-simple element), hence $[L, L] = \Omega_4^+(q)$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$ or to $\mathrm{SU}_2(q)$ and $\mathrm{SL}_2(q)$ and $\mathrm{SU}_2(q)$ have non-abelian Sylow 2-subgroups. □

In the next sections, we will analyse closely the structure of centralisers of semi-simple elements in classical groups. The following elementary lemma will be useful in this context. In what follows, $\mathrm{O}_0^\pm(q)$ are to be interpreted as the trivial group, and $\mathrm{O}_1(q)$ as a cyclic group of order 2. Also note that the center of L is a 2-group.

Lemma 11.2. *Let $t \geq 0$, let d_i, m_i, $1 \leq i \leq t$, be positive integers and let m_0 be a non-negative integer. Let*

$$H = \prod_{0 \leq i \leq t} H_i$$

be a subgroup of L such that H_0 is one of the groups $\mathrm{Sp}_{2m_0}(q)$, $\mathrm{O}_{2m_0+1}(q)$ or $\mathrm{O}_{2m_0}^\pm(q)$ and H_i is isomorphic to $\mathrm{GL}_{m_i}(q^{d_i})$ or $\mathrm{GU}_{m_i}(q^{d_i})$ for $1 \leq i \leq t$. Let Z be a central subgroup of L contained in H such that if the above decomposition of H has more than one non-trivial factor, then $Z \cap H_i = 1$ for all i, $0 \leq i \leq t$. Let T be a Sylow 2-subgroup of H and set $P = (T \cap [L, L]Z)/Z$. Suppose that P is abelian. Then,

(i) $m_i \leq 2$, $0 \leq i \leq t$.

(ii) If $m_0 = 2$, then $H = H_0 = \mathrm{O}_4^+(q)$.

(iii) If $m_i = 2$ for some $i \geq 1$, then $t = 1$ and $m = 0$, that is $H = H_1$. Further, d_1 is odd and $q \equiv \pm 3 \pmod 8$.

(iv) If $H_0 = \mathrm{Sp}_{2n}(q)$ and if $m_0 \neq 0$, then $t = 0$, that is $H = H_0$.

Proof. For $0 \leq i \leq t$, $[H_i, H_i] \leq H \cap [L, L]$, and hence

$$[H_i, H_i]/[H_i, H_i] \cap Z \cong [H_i, H_i]Z/Z \leq (H \cap [L, L]Z)/Z.$$

Suppose first that two of the factors of H are non-trivial, say H_i and H_j, $i \neq j$. Then, by assumption $[H_i, H_i] \cap Z = 1$. It follows from the above that $[H_i, H_i]$ is a subgroup of $(H \cap [L, L])/Z$. In particular, the Sylow 2-subgroups of $[H_i, H_i]$ are abelian. But by Lemma 11.1, $[H_i, H_i]$ has non-abelian Sylow 2-subgroups if $m_i \geq 2$. This proves the first assertions of (ii) and (iii).

Now supppose that some $m_i \geq 2$. By what has just been proved, either $t = 0$ or $t = 1$ and $m_0 = 0$. If $t = 0$, then $H = H_0$ is an orthogonal or symplectic group of dimension m_0. By Lemma 11.1, it follows that $H_0 = \mathrm{O}_4^+(q)$, proving (ii). If $t = 1$ and $m_0 = 0$, then $[H, H]/Z([H, H])$ is a projective special linear or unitary group, and by Lemma 11.1 (i), $m_1 = 2$, $H \cong \mathrm{GL}_2(q^{d_1})$, and $q^{d_1} \equiv \pm 3 \pmod 8$. The last can only hold if d_1 is odd and $q \equiv \pm 3 \pmod 8$. This proves (iii). The proof of (iv) is similar, using the fact that symplectic groups $\mathrm{Sp}_{2n}(q)$ are perfect for $n \geq 2$. □

12. Type A in odd characteristic

By results of Blau and Ellers in [6], Brauer's height zero conjecture holds for all blocks in non-defining characteristic of quasi-simple groups of type A and ${}^2A$, and hence so does Alperin's weight conjecture for all blocks of these groups in odd characteristic with an elementary abelian defect group of order 8, by Landrock's results quoted in Proposition 3.2. This proves in particular Theorem 1.1 for these groups. For future reference, and using methods similar to those in [6], we prove in this and the following section that elementary abelian 2-defect groups of odd rank at least three do not occur in type A and type ${}^2A$, and that blocks with an elementary abelian 2-defect group of even rank at least four of these groups satisfy Alperin's weight conjecture. For the remainder of the paper, we assume $p = 2$.

Theorem 12.1. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G) \cong \mathrm{PSL}_n(q)$ for some positive integer n and some odd prime power q. Let b be a block of kG with an elementary abelian defect group of order 2^r for some integer $r \geq 3$. Then r is even and b satisfies Alperin's weight conjecture.*

Lemma 12.2. *Let n, m, d be positive integers such that $n = md$. Consider $\mathrm{GL}_m(q^d)$ as subgroup of $\mathrm{GL}_n(q)$ through some $\mathbb{F}_q$-decomposition $(\mathbb{F}_q)^n \cong (\mathbb{F}_{q^d})^m$. Denote by $\det$ the determinant function on $\mathrm{GL}_n(q)$. Then there is an element $x \in \mathrm{GL}_m(q^d)$ such that $\det(x)$ has order $q-1$.*

Proof. Let λ be a generator of $\mathbb{F}_{q^d}^\times$ and $f \in \mathbb{F}_q[X]$ the minimal polynomial of λ over $\mathbb{F}_q$. Then f has degree d, and the roots of f are λ, λ^q, λ^{q2},..., $\lambda^{q^{d-1}}$. Let $y \in \mathrm{GL}_d(q)$ with minimal polyomial f. Define $x \in \mathrm{GL}_n(q)$ via $d \times d$-block diagonal matrices where the first block is y and the remaining m blocks are the identity matrices Id_d. Then, in $\mathrm{GL}_n(\bar{\mathbb{F}}_q)$, the element x is conjugate to a diagonal matrix whose diagonal entries are λ, λ^q, λ^{q2},...,$\lambda^{q^{d-1}}$, 1, 1, ...,1. Thus the determinant of x is $\det(x) = \lambda \cdot \lambda^q \cdots \lambda^{q^{d-1}} = \lambda^{\frac{q^d-1}{q-1}}$. Since λ has order q^d-1 it follows that $\det(x)$ has order $q-1$. □

Lemma 12.3. *Let n, m, d be positive integers such that $n = md$. Consider $\mathrm{GL}_m(q^d)$ as subgroup of $\mathrm{GL}_n(q)$ through some $\mathbb{F}_q$-decomposition $(\mathbb{F}_q)^n \cong (\mathbb{F}_{q^d})^m$. If $m \geq 2$ and $q^d \equiv 1 \pmod 4$ then there is an element $y \in \mathrm{GL}_m(q^d)$ of order 4 such that the image of y in $\mathrm{PGL}_n(q)$ has order 4. If moreover $m \geq 3$ we can choose such an element y in $\mathrm{SL}_n(q)$.*

Proof. Since $m \geq 2$, the group $\mathrm{GL}_m(q^d)$ contains a subgroup isomorphic to $\mathbb{F}_{q^d}^\times \times \mathbb{F}_{q^d}^\times$; choose $y = (y_1, 1)$ in this subgroup, where $y_1 \in \mathbb{F}_{q^d}^\times$ has order 4. Then y has an eigenvalue 1, hence if some power y^r is a scalar multiple of Id_n then $y^r = \mathrm{Id}_n$, which shows that the order of y remains unchanged upon taking its image in $\mathrm{PGL}_n(q)$. If $m \geq 3$ then $\mathrm{GL}_m(q^d)$ contains a subgroup isomorphic to $(\mathbb{F}_{q^d}^\times)^3$; choose $y = (y_1, 1, y_3)$ with y_3 such that $\det(y_3) = \det(y_1)^{-1}$, which is possible thanks to Lemma 12.2, and as before, y has the required properties. □

Since the case $\mathrm{PSL}_2(9) \cong A_6$ is dealt with in §7 in order to prove Theorem 12.1 we may assume that $G = \mathrm{SL}_n(q)/Z_+$, where Z_+ is the Sylow 2-subgroup of $Z(\mathrm{SL}_n(q))$. Note that $|Z_+|$ is equal to the 2-part $(n, q-1)_+$ of $(n, q-1)$.

Let b be a block of kG and denote by P a defect group of b. Since $\mathrm{PSL}_n(q) \cong G/Z_-$, where Z_- is the complement of Z_+ in $Z(SL_n(q))$ identified to its image in G, the image of P in $\mathrm{PSL}_n(q)$ is isomorphic to P. Since Z_+ is a central 2-subgroup, b is the image of a unique block $\tilde{b}$ of $k\,\mathrm{SL}_n(q)$, and the inverse image $\tilde{P}$ of P in $\mathrm{SL}_n(q)$ is a defect group of $\tilde{b}$. Let d be a block of $k\,\mathrm{GL}_n(q)$ covering $\tilde{b}$ with a defect group T such that $T \cap \mathrm{SL}_n(q) = \tilde{P}$. By [24, Proposition 6.3] the block $\tilde{b}$ is stable under the 2-part of $\mathrm{GL}_n(q)/\mathrm{SL}_n(q)$, and hence $T/\tilde{P}$ is cyclic of order the 2-part $(q-1)_+$ of $q-1$. By [12, Théorème 3.3], there exists a semi-simple element s of odd order in $\mathrm{GL}_n(q)$ such that T is a Sylow 2-subgroup of $C_{\mathrm{GL}_n(q)}(s)$ (this can also be seen as a consequence of the Jordan decomposition of Theorem 10.1). Further there exist positive integers m_i, d_i, $1 \leq i \leq t$ such that

$$n = \sum_{1 \leq i \leq t} m_i d_i,$$

and setting $H_i = \mathrm{GL}_{m_i}(q^{d_i})$, there is a decomposition

$$C_{\mathrm{GL}_n(q)}(s) \equiv \prod_{1 \leq i \leq t} H_i$$

corresponding to a subspace decomposition of the underlying $\mathbb{F}_q$-vector vector space as isotypic $\mathbb{F}_q[s]$-modules. In particular, $H_i = \mathrm{GL}_{m_i}(q^{d_i})$ is a subgroup of $\mathrm{GL}_{m_i d_i}(q)$ through some $\mathbb{F}_q$-decomposition $(\mathbb{F}_q)^{m_i d_i} \cong (\mathbb{F}_{q^{d_i}})^{m_i}$.

Lemma 12.4. *With the notation above, suppose that P is elementary abelian and that $t \geq 3$. Then the following holds.*

(i) $q \equiv 3 \pmod 4$.

(ii) $m_i = 1$ *for* $1 \leq i \leq t$.

(iii) d_i *is odd for* $1 \leq i \leq t$.

(iv) If t is even then $|P| = 2^{t-2}$, and if t is odd then $|P| = 2^{t-1}$; in particular, the 2-rank of P is even.

(v) The block d of $\mathrm{GL}_n(q)$ is nilpotent with an elementary abelian defect group T of order 2^t.

(vi) The block b of kG satisfies Alperin's weight conjecture.

Proof. Clearly, $P = (T \cap \mathrm{SL}_n(q))/Z_+$ and $Z_+ \cap H_i = 1$ for any i such that $H_i \neq C_L(s)$. Thus Lemma 11.2 (ii) and (iii) apply with $L = \mathrm{GL}_n(q)$, $H = C_L(s)$ and $Z = Z_+$, and assertion (ii) is immediate. If $q \equiv 1 \pmod 4$ or if d_1 is even, then the group $H_1 = \mathrm{GL}_1(q^{d_1})$ contains an element y_1 of order 4. By Lemma 12.2 the group $H_2 = \mathrm{GL}_1(q^{d_2})$ contains a 2-element y_2 such that $\det(y_2) = \det(y_1)^{-1}$. Thus $x = y_1 y_2 \in \mathrm{SL}_n(q) = [L, L]$. Since $t \geq 3$, we have $H_1 \times H_2 \cap Z_+ = 1$, and it follows that the image xZ_+ of x in P has order 4, a contradiction. Thus (i) and (iii) hold. Hence T is a Sylow 2-subgroup of $\prod_{i=1}^{t} \mathbb{F}_{q^{d_i}}^{\times}$. Since the d_i are odd and $q \equiv 3 \pmod 4$ this implies that T is elementary abelian of rank t, and hence $\tilde{P}$ is elementary abelian of rank $t-1$. If n is odd then $P \cong \tilde{P}$ and since $n = \sum_{i=1}^{t} d_i$ and the d_i are odd it follows that t is odd, hence $|P| = 2^{t-1}$. If n is even then $|P| = \frac{|\tilde{P}|}{2}$ and since $n = \sum_{i=1}^{t} d_i$ and the d_i are odd it follows that t is even and $|P| = 2^{t-2}$, which proves (iv). Since T is clearly abelian, (v) is immediate from Corollary 10.2. Statement (vi) follows from (v) and Proposition 2.4. □

Lemma 12.5. *Suppose that P is elementary abelian and that $t = 2$. Then $m_1 = m_2 = 1$, $|P| \leq 4$ and $\tilde{P}$ contains an element of order $max((q^{d_1} - 1)_+, (q^{d_2} - 1)_+)$.*

Proof. The fact that $m_1 = m_2 = 1$ is a consequence of Lemma 11.2. So $H_i \cong \mathrm{GL}_1(q^{d_i})$. By Lemma 12.2, it follows that $\tilde{P} = T \cap \mathrm{SL}_n(q)$ contains elements of order $max((q^{d_1} - 1)_+, (q^{d_2} - 1)_+)$. □

Lemma 12.6. *Suppose that P is elementary abelian and that $t = 1$. Then $m_1 \leq 2$.*

If $m_1 = 2$, then d_1 is odd, $n = 2d_1$, $|P| = 4$ and $\tilde{P}$ is a quaternion group of order 8.

If $m_1 = 1$, then T, $\tilde{P}$ and P are cyclic, $|P| = 2$ if and only if n is even, and either $q \equiv 3 \pmod 4$ and $n_+ \leq (q-1)_+$ or $q \equiv 1 \pmod 4$ and $n_+ = 2(q-1)_+$.

Proof. By Lemma 11.2, $m_1 \leq 2$. Suppose that $m_1 = 1$, so T is a Sylow 2-subgroup of $\mathrm{GL}_1(q^n)$ and in particular, T is a cyclic group of order $(q^n - 1)_+$. It follows from Lemma 12.2 that $\tilde{P} = P \cap \mathrm{SL}_n(q)$ is cyclic of order $\frac{(q^n-1)_+}{(q-1)_+}$, and hence P is cyclic of order $\frac{(q^n-1)_+}{(q-1)_+|Z_+|}$. Now, $|Z_+| = min(n_+, (q-1)_+)$. If n is odd then $(q^n - 1)_+ = (q-1)_+$ and if n is even then $(q^n - 1)_+ = \frac{(q^2-1)_+ n_+}{2}$. The statement of the lemma for the case $m_1 = 1$ follows by an easy calculation. Now suppose $m_1 = 2$. Then by Lemma 11.2, (ii) we have $C_{\mathrm{GL}_n(q)}(s) = \mathrm{GL}_2(q^{d_i})$, $q \equiv \pm 3 \pmod 8$, d_i is odd and $n = 2d_i$. With this arithmetic, one sees easily that $|P| = 4$ and $|\tilde{P}| = 8$. Finally, $\tilde{P}$ is quaternion since it contains a subgroup isomorphic to the Sylow 2-subgroups of $\mathrm{SL}_2(q^{d_i})$, which by Lemma 11.1 are quaternion. □

Proof of Theorem 12.1. If $t \geq 3$, Lemma 12.4 shows that the 2-rank of P is at least 4 and even and that b satisfies Alperin's weight conjecture. If $t \leq 2$, Lemma 12.5 and Lemma 12.6 show that the rank of P is at most 2. □

We also note the following.

Lemma 12.7. *Suppose that $n = 2m$, $m \geq 1$, and P is elementary abelian of order 2 or 4. Then the inverse image of P in a non-split central extension $2.G$ has an element of order 4 unless $t = 4$, $q \equiv 3 (\mathrm{mod}\ 4)$, and d_i is odd for $1 \leq i \leq 4$. In particular, if P has order 2, then the inverse image has order 4.*

Proof. The central extension $2.G$ may be assumed to be a central quotient of $\mathrm{SL}_n(q)$ and the inverse image, P_0 of P in $2.G$ is a quotient of $\tilde{P}$ by a cyclic (central) group of order $\frac{1}{2}|Z_+|$. We will show that unless we are in the exceptional case above, that either $\tilde{P}$ is cyclic or that $\tilde{P}_0$ contains an element of order $2|Z|_+$. If $t = 1$ and $m_1 = 1$, then $C_L(s)$, and hence $\tilde{P}$ is cyclic. So certainly P_0 is cyclic. If $t = 1$ and $m_1 = 2$, then $[H_2, H_2] \leq C_L(s) \cap [L, L]$ is a special linear group of dimension 2 and hence $\tilde{P} \cap [H_2, H_2]$ is a quaternion group of order $q^{2d} - 1$. In particular, $\tilde{P}$ contains an element of order $\frac{1}{2}(q^{2d} - 1)$. Since $|Z|_+ \leq (q-1)_+$, if d is even, then $\tilde{P}$ contains an element of order at least $2|Z|_+$. So we may assume that d is odd. Then $|Z|_+ = 2$, and $\frac{1}{2}(q^{2d} - 1) \geq 4 = 2|Z|_+$. Now suppose $t = 2$. So, $m_1 = m_2 = 1$ and $n = d_1 + d_2$. Thus, $T = T_1 \times T_2$, with H_i a cyclic group of order $(q^{d_i} - 1)_+$, $i = 1, 2$. We assume without loss of generality that $|T_1| \geq |T_2|$. By Lemma 12.2, it is easy to see that $\tilde{P} = T \cap \mathrm{SL}_n(q)$ is a direct product $Q_1 \times Q_2$ such that Q_1 is cyclic of order $|T_1|$ and Q_2 is cyclic of order $\frac{|T_2|}{(q-1)_+}$. If d_1 is even, then $|T_1| \geq 2(q-1)_+ \geq 2|Z|_+$. If d_1 is odd, then d_2 is also odd (as n is even), hence $T_2 = 1$, that is $\tilde{T}$ is cyclic. Finally, suppose that $t \geq 3$. By Lemma 12.4, P is not of order 2, and P is of order 4 if and only if $t = 4$, $q \equiv 3 (\mathrm{mod}\ 4)$, and d_i is odd for $1 \leq i \leq 4$. Note that in this case, $\tilde{P}$ is elementary abelian of order 8. □

13. Type ^2A in odd characteristic

We show that type ^2A yields no blocks with elementary abelian defect groups of order 8; in fact, more generally, we have the following result:

Theorem 13.1. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G) \cong \mathrm{PSU}_n(q)$ for some positive integer n and some odd prime power q. Let b be a block of kG with an elementary abelian defect group of order 2^r for some integer $r \geq 3$. Then r is even and b satisfies Alperin's weight conjecture.*

The proof of this follows the same lines as the untwisted case. We give details for the convenience of the reader. We single out two elementary observations which we will use in the proof below:

Lemma 13.2. *Let n, m, d be positive integers and denote by* det *the determinant function on* $\mathrm{GL}_n(\bar{\mathbb{F}}_q)$.

(i) Suppose that $n \geq 2d$. Consider the inclusions $\mathrm{GL}_1(q^{2d}) \leq \mathrm{GL}_d(q^2) \leq \mathrm{GU}_{2d}(q) \leq \mathrm{GU}_n(q)$, where $\mathrm{GL}_1(q^{2d})$ is a subgroup of $\mathrm{GL}_d(q^2)$ through some $\mathbb{F}_{q^2}$-vector space isomorphism $(\mathbb{F}_{q^2})^d \cong \mathbb{F}_{q^{2d}}$, $GL_d(q^2)$ is a subgroup of $\mathrm{GU}_{2d}(q)$ through some $\mathbb{F}_{q^2}$-vector space embedding $(\mathbb{F}_{q^2})^d \hookrightarrow (\mathbb{F}_{q^2})^d \oplus (\mathbb{F}_{q^2})^d$ of the form $\lambda \to \lambda + \lambda^{-q}$ and $\mathrm{GU}_{2d}(q)$ is a subgroup of $\mathrm{GU}_n(q)$ through some decomposition $(\mathbb{F}_{q^2})^n \cong (\mathbb{F}_{q^2})^{2d} \oplus (\mathbb{F}_{q^2})^{n-2d}$. There is an element $x \in \mathrm{GL}_1(q^{2d})$ such that $\det(x)$ has order $q+1$.

(ii) Suppose that d is odd and that $n \geq d$. Consider the inclusions $\mathrm{GU}_1(q^d) \leq \mathrm{GU}_d(q) \leq \mathrm{GU}_n(q)$, where $\mathrm{GU}_1(q^d)$ is a subgroup of $\mathrm{GU}_d(q)$ through an irreducible unitary representation of $\mathrm{GU}_1(q^d)$ on a d-dimensional $\mathbb{F}_{q^2}$-space, and where $\mathrm{GU}_d(q)$ is a subgroup of $\mathrm{GU}_n(q)$ through some decomposition $(\mathbb{F}_{q^2})^n \cong (\mathbb{F}_{q^2})^{2d} \oplus (\mathbb{F}_{q^2})^{n-2d}$. There is an element $x \in \mathrm{GL}_1(q^{2d})$ such that $\det(x)$ has order $q+1$.

Proof. (i) Let λ be a generator of $\mathbb{F}_{q^{2d}}^\times$ and $f \in \mathbb{F}_{q^2}[X]$ the minimal polynomial of λ over $\mathbb{F}_{q^2}$. Then f has degree d, and the roots of f are λ, λ^{q^2}, λ^{q^4},..., $\lambda^{q^{2(d-1)}}$. Let $x \in \mathrm{GL}_d(q^2)$ with minimal polyomial f. Then, in $\mathrm{GL}_n(\bar{\mathbb{F}}_q)$, the element x is conjugate to a diagonal matrix whose diagonal entries are λ, λ^{q^2}, λ^{q^4},...,$\lambda^{q^{2(d-1)}}$, λ^{-q}, $(\lambda^{q^2})^{-q}$, $(\lambda^{q^4})^{-q}$,...,$(\lambda^{q^{2(d-1)}})^{-q}$, $1\cdots 1$. Thus the determinant of x is $\det(x) = aa^{-q}$, where $a = \lambda \cdot \lambda^{q^2} \cdots \lambda^{q^{2(d-1)}} = \lambda^{\frac{q^{2d}-1}{q^2-1}}$. Since λ has order $q^{2d}-1$ it follows that $\det(x)$ has order $q+1$.

(ii) Let λ be an element of order q^{d+1} in $\mathbb{F}_{q^{2d}}^\times$ and $f \in \mathbb{F}_{q^2}[X]$ the minimal polynomial of λ over $\mathbb{F}_{q^2}$. Since d is odd, f has degree d. Let $x \in \mathrm{GU}_d(q)$ with minimal polyomial f. In $\mathrm{GL}_n(\bar{\mathbb{F}})$, the element x is conjugate to a diagonal matrix whose diagonal entries are λ, λ^{q^2}, λ^{q^4},...,$\lambda^{q^{2(d-1)}}$. Thus the determinant of x is

$$\det(x) = \lambda \cdot \lambda^{q^2} \cdots \lambda^{q^{2(d-1)}} = \lambda^{\frac{q^{2d}-1}{q^2-1}}.$$

Since λ has order q^d+1 and since d being odd, the integers, $\frac{q^d-1}{q-1}$ and q^d+1 are relatively prime, it follows that $\det(x)$ has order $q+1$. □

We turn now towards the proof of Theorem 13.1. Since the case $\mathrm{PSU}_4(3)$ is dealt with in §7 in order to prove Theorem 12.1 we may assume that $G = \mathrm{SU}_n(q)/Z_+$, where Z_+ is the Sylow 2-subgroup of $Z(\mathrm{SU}_n(q))$. Note that $|Z_+|$ is equal to the 2-part $(n, q+1)_+$ of $(n, q+1)$. Let b be a block of kG and denote by P a defect group of b. Since $\mathrm{PSU}_n(q) \cong G/Z_-$, where Z_- is the complement of C_+ in $Z(SL_n(q))$ identified to its image in G, the image of P in $\mathrm{PSU}_n(q)$ is isomorphic to P. Since Z_+ is a central 2-subgroup, b is the image of a unique block $\tilde{b}$ of $k\,\mathrm{SU}_n(q)$, and the inverse image $\tilde{P}$ of P in $\mathrm{SU}_n(q)$ is a defect group of $\tilde{b}$. Let d be a block of $k\,\mathrm{GU}_n(q)$ covering $\tilde{b}$ with a defect group T such that $T \cap \mathrm{SU}_n(q) = \tilde{P}$. By [24, Proposition 6.3] the block $\tilde{b}$ is stable under the 2-part of $\mathrm{GU}_n(q)/\mathrm{SU}_n(q)$, and hence $T/\tilde{P}$ is cyclic of order the 2-part $(q+1)_+$ of $q+1$. By Fong-Srinivasan [35], T is isomorphic to a Sylow 2-subgroup of

$$H = \prod_{i=1}^{s} \mathrm{GL}_{m_i}(q^{2d_i}) \times \prod_{j=1}^{t} \mathrm{GU}_{n_j}(q^{e_j})$$

for some non-negative integers s, t, and some positive integers m_i, n_i d_i, e_j such that all e_j are odd and satisfy

$$n = \sum_{i=1}^{s} 2m_i d_i + \sum_{j=1}^{t} n_j e_j.$$

We keep this notation for the remainder of this section.

Lemma 13.3. *(i) Suppose that $\sum_{1\leq i\leq s} m_i + \sum_{1\leq j\leq t} n_j \geq 3$. If for some i, j, either $\mathrm{GL}_1(q^{2d_i})$ or $\mathrm{GU}_1(q^{e_j})$ contains an element of order 2^a, then so does P.*

(ii) Suppose that $\sum_{1\leq i\leq s} m_i + \sum_{1\leq j\leq t} n_j \geq 4$ and either $m_i \geq 2$ for some i, $1 \leq i \leq s$ or $n_j \geq 2$ for some j, $1 \leq j \leq t$. Then P contains an element of order 8.

(iii) Suppose that $n_j \geq 2$ for some j, $1 \leq j \leq t$ and $\sum_{1\leq i\leq s} m_i + \sum_{1\leq j\leq t} n_j \geq 3$. Then, $\tilde{P}$ contains an element of order 8.

Proof. (i) This follows easily from Proposition 13.2 using arguments similar to Lemma 12.3.

(ii) Similar argument to (i) using the fact that $\mathrm{GL}_2(q')$ and $\mathrm{GU}_2(q')$ contain elements of order 8 for any odd prime power q'.

(iii) Again, we use the fact that $\mathrm{GU}_2(q^{e_i})$ has an element of order 8. □

Lemma 13.4. *Suppose that P is elementary abelian and that $s + t \geq 3$. Then the following hold.*

(i) $s = 0$.

(ii) $n_i = 1$ for all i, $1 \leq i \leq t$.

(iii) $q \equiv 1 \pmod 4$.

(iv) If t is even then $|P| = 2^{t-2}$, and if t is odd then $|P| = 2^{t-1}$; in particular, the 2-rank of P is even.

(v) The block d of $\mathrm{GU}_n(q)$ is nilpotent with an elementary abelian defect group T of order 2^s.

(vi) The block b of kG satisfies Alperin's weight conjecture.

Proof. Since $\mathrm{GL}_1(q^{2d_1})$ contains an element of order 8, conclusion (i) is immediate from Lemma 13.3(i). Assume from now on that $s = 0$. If $n_1 \geq 2$, then the fact that $s + t \geq 3$ implies that $\sum_{1\leq j\leq s} n_j \geq 4$. So, by Lemma 13.3(ii), P has an element of order 8, a contradiction. So, (ii) holds. If $q \equiv 3 \pmod 4$, then $\mathrm{GU}_1(q^{e_i})$ contains an element of order 4, and again by Lemma 13.3(i), P has an element of order 4. This proves (iii). Hence T is a Sylow 2-subgroup of $\prod_{j=1}^{t} \mathbb{F}_{q^{e_j}}^{\times}$. Since the e_j are odd and $q \equiv 1 \pmod 4$ this implies that T is elementary abelian of rank t, and hence $\tilde{P}$ is elementary abelian of rank $t-1$. If n is odd then $P \cong \tilde{P}$ and since $n = \sum_{j=1}^{t} e_j$ and the e_j are odd it follows that t is odd, hence $|P| = 2^{t-1}$. If n is even then $|P| = \frac{|\tilde{P}|}{2}$ and since $n = \sum_{j=1}^{t} e_j$ and the e_j are odd it follows that t is even and $|P| = 2^{t-2}$, which proves (iv). Finally, the block d of $\mathrm{GU}_n(q)$ is nilpotent in that case because (ii) implies that the centraliser of the semi-simple element labelling d is a torus, whence (v). Statement (vi) follows from (v) and Proposition 2.4. □

Lemma 13.5. *Suppose that P is elementary abelian and that $s + t \leq 2$. Then $|P| \leq 4$.*

Proof. Suppose $s+t = 1$; that is, T is a Sylow 2-subgroup of $\mathrm{GL}_m(q^{2d})$, where $m = m_1$ and $d = d_1$, and we have $n = 2md$ or T is a Sylow 2-subgroup of $\mathrm{GU}_m(q^e)$, where $m = n_1$ and $e = e_1$, and we have $n = me$. In the former case, if $m = 1$ then T is cyclic, hence $|P| \leq 2$, so we may assume $m \geq 2$. If $m \geq 3$, then since $\mathrm{GL}_1(q^{2d})$ contains an element of order 8, by Lemma 13.3 (i), P has an element of order 8, a contradiction. This contradiction shows $m = 2$. Thus $n = 4d$, and so T is a Sylow 2-subgroup of $\mathrm{GL}_2(q^{2d})$, and one easily checks that any elementary abelian subquotient of T has rank at most 2. In the latter case, again if $m = 1$, then T is cyclic. If $m \geq 4$, then by Lemma 13.3 (ii), P contains an element of order 8, a contradiction. If $m = 3$, then by Lemma 13.3 (iii), $\tilde{P}$ has an element of order 8. But $n = 3e$ is odd, and hence $P \equiv \tilde{P}$. This contradiction shows that $m = 2$, and T is a Sylow 2-subgroup of $\mathrm{GU}_2(q^2)$. But any elementary abelian subquotient of T has rank at most 2. Now suppose that $s + t = 2$ and $\sum_i m_i + \sum_j n_j \geq 3$. By Lemma 13.3 (i), $s = 0$, whence $t = 2$. By Lemma 13.3 (ii), at least one of n_1 or n_2 equals 1, say $n_2 = 1$. If $n_1 \geq 3$, then by Lemma 13.3 (ii), P contains an element of order 8. I If $n_2 = 2$, then by Lemma 13.3 (iii), $\tilde{P}$ contains an element of order 8. But in this case, $n = 2e_1 + e_2$ is odd, which means that $\tilde{P} \equiv P$, and hence P has an element of order 8. If $n_1 = 1$, then T is abelian of rank 2, hence P has order at most 4. Finally suppose that $s + t = 2$ and $\sum_i m_i + \sum_j n_j = 2$. If $s = 0, t = 2$, then $n_1 = n_2 = 1$; if $s = 1, t = 1$, then $m_1 = n_1 = 1$ and if $s = 2, t = 0$, then $m_1 = m_2 = 1$. In all cases T is abelian of rank 2 whence P has order at most 4. □

Proof of Theorem 13.1. This is immediate from the preceding lemmas. □

14. Orthogonal and symplectic groups in odd characteristic

We show that blocks of orthogonal and symplectic groups with elementary abelian defect groups of order at least 8 are all nilpotent.

Theorem 14.1. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is isomorphic to one of $\mathrm{PSp}_{2n}(q)$, $n \geq 2$, $\mathrm{P}\Omega_{2n+1}(q)$, $n \geq 3$ or $\mathrm{P}\Omega^{\pm}_{2n}(q)$, $n \geq 4$ for some odd prime power q. Let b be a block of kG with elementary abelian defect groups of order 2^r for some integer $r \geq 3$. Then b is nilpotent.*

Notation. The group G will denote one of the groups in the above theorem. Further, we define L, L_0 and $\tilde{G}$ as follows.

If $G/Z(G) = \mathrm{PSp}_{2n}(q)$, then $L = L_0 = \mathrm{Sp}_{2n}(q)$ and $\tilde{G} = \mathrm{Sp}_{2n}(q)$.

If $G/Z(G) = \mathrm{P}\Omega_{2n+1}(q)$, then $L = \mathrm{O}_{2n+1}(q)$, $L_0 = \mathrm{SO}_{2n+1}(q)$ and $\tilde{G} = \Omega_{2n+1}(q)$.

If $G/Z(G) = \mathrm{P}\Omega^{+}_{2n}(q)$, then $L = \mathrm{O}^{+}_{2n}(q)$, $L_0 = \mathrm{SO}^{+}_{2n}(q)$ and $\tilde{G} = \Omega^{+}_{2n}(q)$.

If $G/Z(G) = \mathrm{P}\Omega^{-}_{2n}(q)$, then $L = \mathrm{O}^{-}_{2n}(q)$, $L_0 = \mathrm{SO}^{-}_{2n}(q)$ and $\tilde{G} = \Omega^{-}_{2n}(q)$.

So, $\tilde{G} \lhd L_0 \lhd L$ with the indices of the inclusions being 1 or 2 and $G = \tilde{G}/Z$ where Z is a central subgroup of $\tilde{G}$ of order 1 or 2. Let b be a block of kG and denote by P a defect group of b. Since Z is a central 2-subgroup, b is the image of a unique block $\tilde{b}$ of $k\tilde{G}$, and the inverse image $\tilde{P}$ of P in $k\tilde{G}$ is a defect group of $\tilde{b}$. Let d_0 be a block of kL_0 covering $\tilde{b}$ with a defect group T_0 such that $T_0 \cap \tilde{G} = \tilde{P}$. We note that the block $\tilde{b}$ is L_0-stable (see [24, Corollary 6.4]) whence, $d_0 = \tilde{b}$ and $T_0/\tilde{P}$ is cyclic of order $[L_0 : \tilde{G}]$. Let d be a block of kL covering $d_0 = \tilde{b}$ and T a defect group of b containing T_0. As a byproduct of the Jordan decomposition of blocks (see Theorem 10.1), there exists a semi-simple element $s \in L_0$ of odd order such that T_0 is a Sylow 2-subgroup of $C_{L_0}(s)$ and T is a Sylow 2-subgroup of $C_L(s)$ (see [2, (5A)]).

Lemma 14.2. *If the defect groups of kL_0d_0 are abelian, then kL_0d_0, kLd, $k\tilde{G}\tilde{b}$ and kGb are all nilpotent blocks.*

Proof. $L_0 = \mathbf{L}_0^F$, where $\mathbf{L}_0$ is a simple algebraic group of type B, C or D over $\bar{\mathbb{F}}_q$ and $F : \mathbf{L}_0 \to \mathbf{L}_0$ is a Frobenius endomorphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{L}_0$. Thus, by Corollary 10.2,

$kL_o d_0$ is nilpotent. Since $\tilde{G}$ is a normal subgroup of L_0 of index a power of 2 and d_0 covers $\tilde{b}$, $k\tilde{G}\tilde{b}$ is nilpotent. Since G is a quotient of $\tilde{G}$ by a central 2-subgroup, and $\tilde{b}$ lifts b, kGb is nilpotent. Finally L_0 is of index 1 or 2 in L, and d covers d_0, hence kLd is nilpotent. □

For the rest of this section s will denote a semi-simple element of odd order in L_0 such that T_0 is a Sylow 2-subgroup of $C_{L_0}(s)$ and T is a Sylow 2-subgroup of $C_L(s)$.

Lemma 14.3. *Suppose that $G = \mathrm{PSp}_{2n}(q)$, $n \geq 2$. If P is abelian, then b is nilpotent.*

Proof. Note that $\tilde{G} = [L, L] = L_0 = L$, $G = L/Z$, $|Z| = 2$, and $d = d_0 = \tilde{b}$. Further, by [36, §1], and noting that s has odd order, the group $H = C_L(s)$ is a direct product of groups H_i as in Lemma 11.2 with $H_0 \cong \mathrm{Sp}_{2m_0}(q)$ and

$$n = m_0 + \sum_{1 \leq i \leq t} m_i d_i.$$

The above decomposition of $C_L(s)$ corresponds to the orthogonal decomposition of the underlying symplectic space as a direct sum of isotypic $\mathbb{F}_q[s]$-modules (for instance, H_0 corresponds to the 1-eigen space of s). In particular, if the decomposition has more than one non-trivial factor, then $Z \cap H_i = 1$ for all i, $0 \leq i \leq t$. We have $T = T_0 = \tilde{P}$ and $P = T \cap [L, L]/Z = T/Z$. Suppose that P is abelian. If $m_0 \neq 0$, then by Lemma 11.2(ii) and (iv), $m_0 = 1$, $H = H_0 = \mathrm{Sp}_2(q)$. In particular, $n = m_0 = 1$, a contradiction. Thus $m_0 = 0$. Suppose that $m_i = 2$ for some $i \geq 1$. Then by Lemma 11.2, $t = 1$ and $H = \mathrm{GL}_2(q^{d_1})$ or $H_1 = \mathrm{GU}_2(q^{d_1})$. But as observed above, P is the quotient of a Sylow 2-subgroup of H by $Z(H)$, and $\mathrm{PGL}_2(q')$ and $\mathrm{PGU}_2(q')$ have non-abelian Sylow 2-subgroups for any odd prime power q', a contradiction. Thus $m_0 = 0$ and $m_i \leq 1$ for all i, $1 \leq i \leq t$. In particular, H and therefore T_0 is abelian. The result follows by Lemma 14.2. □

Before we proceed, we recall the structure of $C_L(s)$ when L is an orthogonal group as described in [36, §1]. Let V be an underlying $\mathbb{F}_q$-vector space for L, and let $\tau : V \to \mathbb{F}_q$ be a non-degenerate quadratic form underlying L. So $L = I(V)$, the subgroup of $\mathrm{GL}(V)$ consisting of isometries with respect to τ, $L_0 = I_0(V)$, the subgroup of $I(V)$ consistsing of matrices of determinant 1 and $\tilde{G} = \Omega(V) = [I(V), I(V)]$. Let V_0 denote the 1-eigen space of s. The space V decomposes as an orthogonal direct sum

$$V = V_0 \oplus (\oplus_{1 \leq i \leq t} V_i)$$

where for $i \geq 1$, V_i, is an isotypic $\mathbb{F}_q[s]$-module, such that V_i and V_j have no common irreducible $\mathbb{F}_q[s]$-summands for $0 \leq i \neq j \leq t$ and such that

$$C_L(s) = \prod_i C_L(s) \cap \mathrm{GL}(V_i).$$

Here $\prod_i \mathrm{GL}(V_i)$ is considered as a subgroup of $\mathrm{GL}(V)$ in the standard way. Further, setting $H_i = C_L(s) \cap \mathrm{GL}(V_i)$, we have that

$$H_i = \begin{cases} L \cap \mathrm{GL}(V_i) & \text{if} \quad i = 0 \\ \mathrm{GL}_{m_i}(\epsilon_i q^{d_i}) & \text{if} \quad i \geq 1. \end{cases}$$

Here for each $i \geq 1$, $2d_i m_i$ is the $\mathbb{F}_q$-dimension of V_i and $H_0 = I(V_0)$. Also, $\epsilon_i \in \{\pm 1\}$ and $\mathrm{GL}_{m_i}(\epsilon_i q^{d_i})$ denotes $\mathrm{GU}_{n_i}(q^{d_i})$ if $\epsilon_i = -1$. If $i \geq 1$, then $H_i \leq L_0$. Thus,

$$C_{L_0}(s) = (H_0 \cap L_0) \times \prod_{1 \leq i \leq s} H_i.$$

For each i, $0 \le i \le t$, the restriction of τ to V_i is non-degenerate. This form has maximal Witt index if $\epsilon_i^{m_i} = 1$ and is of non-maximal Witt index if $\epsilon_i^{m_i} = -1$. For $1 \le i \le t$, let t_i be the unique involution in the centre of H_i. As an element of $\mathrm{GL}(V)$, t_i acts as -1 on V_i and as 1 on all V_j, $j \ne i$. Now $\tilde{G} = [I(V), I(V)]$ is the kernel of the spinorial norm from $I_0(V)$ to $\mathbb{F}_q^\times/\mathbb{F}_q^{\times 2}$ (see [39, §2.7]). From this it follows that $t_i \in \Omega(V)$ if and only if $q^{d_i m_i} \equiv \epsilon_i^{m_i} \pmod 4$. We will use this fact in the sequel.

Lemma 14.4. *Suppose that $G = \mathrm{P}\Omega_{2n+1}(q)$, $n \ge 3$ and that P is elementary abelian with $|P| \ge 8$. Then b is nilpotent.*

Proof. Note that since the dimension of the underlying vector space is odd, $G = \tilde{G} = [L, L]$, $\tilde{G}$ is of index 2 in L_0 and L_0 is of index 2 in L. In particular, $\tilde{P} = P$. By Lemma 11.2, either $m_1 = 2$ and $m_i = 0$ for all i different from 1, or all $m_i \le 1$. In the former case, again by Lemma 11.2, T is isomorphic to a Sylow 2-subgroup of a 2-dimensional general linear group. In particular, T has an element of order 8. Since $T \le L$ in this case, P is of index 2 in of T, hence P has an element of order 4, a contradiction. We assume from now on that all $m_i \le 1$. If $m_0 = 0$, then T and therefore T_0 is abelian and we are done by Corollary 14.2. So, $m_0 = 1$, i.e. $H_0 = \mathrm{O}_3(q)$. Since $n \ge 2$, $i \ge 1$, i.e., $m_1 \ne 0$. Let T^i be the i-th component of T. We claim that $T^i \not\le P$ for any $i \ge 1$. Indeed, suppose the contrary. Then $T^i = \langle t_i \rangle$ has order 2. In particular, this means that $q^{d_i} \not\equiv \epsilon_i \pmod 4$. But since $m_i = 1$, this means that $t_i \notin \Omega(V)$ and hence $t_i \notin P$. This proves the claim. By Lemma 11.2, $T^0 \cap H_0$ is a dihedral group of order at least eight. Also, clearly $T^0 \cap H_0 \le T_0$. Since P is of index 2 in T_0, and since as just shown $T^1 \not\le P$, it follows that P contains a subgroup isomorphic to $T^0 \cap H_0$, an impossibility as P is abelian and $T^0 \cap \mathrm{SO}_3(q)$ is not. $\square$

Lemma 14.5. *Suppose that $G = \mathrm{P}\Omega_{2n}^+(q)$, or $\mathrm{P}\Omega_{2n}^-(q)$, $n \ge 4$ and that P is elementary abelian with $|P| \ge 8$. Then b is nilpotent.*

Proof. We first consider $G = \mathrm{P}\Omega_{2n}^+(q)$. If $m_1 = 2$, then by Lemma 11.2(i) and (iii), $m_j = 0$ for $j \ne i$, $C_L(s) = H_1 \cong \mathrm{GL}(\epsilon_1 q^{d_1})$, where $q \equiv \pm 3 \pmod 8$ and d_1 is odd. But $q^{d_1 m_1} \equiv 1 \equiv \epsilon_1^{m_1} \pmod 4$, hence the central involution t_1 of H_1 is in $[L, L]$ and P is a subgroup of $T/\langle t_i \rangle$. Since $T/\langle t_i \rangle$ is a dihedral group (see [21]), P has rank at most 2, a contradiction. Now suppose $m_0 = 2$. Then, $m_i = 0$ for all $i \ge 1$, and $n = 2$, a contradiction, as n is assumed to be at least 3. Thus, $m_i \le 1$ for all i. If $m_0 = 0$, then T_0 is abelian and we are done by Theorem 14.2. So, suppose that $m_1 = 1$. Then $H_0 = \mathrm{O}_2^\pm(q)$ is a dihedral group and $H_0 \cap L_0 = \mathrm{SO}_2^\pm(q)$ is a cyclic group (see [39, §2.7]). Since H_i is also cyclic for all $i \ge 1$, T_0 is abelian and we are done by Theorem 14.2. The proof for $G = \mathrm{P}\Omega_{2n}^-(q)$ is similar. $\square$

Proof of Theorem 14.1. This is immediate from the preceding lemmas. $\square$

15. Type G_2, ${}^2G_2$ and ${}^3D_4$ in odd characteristic

Let q an odd prime power. The 2-rank of $G_2(q)$, ${}^2G_2(q)$ and ${}^3D_4(q)$ is 3; see e.g. [37, §1], [39, Theorem 4.10.5].

Proposition 15.1. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is simple of type $G_2(q)$. Then kG has no block with an elementary abelian defect group of order 8.*

Proof. This follows from [42], where the 2-blocks of G are determined. Alternatively, one can use the arguments in [24, 12.2]: there is a unique conjugacy class of involutions u in G and by

[39, 4.5.1], $C_G(u) \cong Z(G)\times$ $(2.(\mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q)).2)$. The acting 2-automorphism in the second factor is inner-diagonal, hence stabilises any block of $2.(\mathrm{PSL}_2(q)\times\mathrm{PSL}_2(q))$. Thus a block of $C_G(u)$ with elementary abelian defect group of order 8 would cover a block of $2.(\mathrm{PSL}_2(q)\times\mathrm{PSL}_2(q))$ with a Klein four defect group, whose image modulo the central involution would yield a block of $\mathrm{PSL}_2(q)\times\mathrm{PSL}_2(q)$ with a defect group of order 2. Any block of this direct product is of the form $c_0\otimes c_1$, where c_0, c_1 are blocks of $\mathrm{PSL}_2(q)$, so exactly one of c_0, c_1 would have defect zero and the other defect one. But since any inverse image of an involution in $\mathrm{PSL}_2(q)$ in $\mathrm{SL}_2(q)$ has order 4 this is impossible. □

Proposition 15.2. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is simple of type ${}^2G_2(q)$. The principal block b_0 is the unique block of kG having an elementary abelian defect P group of order 8, and we have $|\mathrm{Irr}_K(G,b_0)| = 8$.*

Proof. The Sylow 2-subgroups of ${}^2G_2(q)$ are elementary abelian of order 8. The simple group of type ${}^2G_2(q)$ has trivial Schur multiplier, hence $Z(G) = 1$. In that case we have $N_G(P) \cong P \rtimes E$, with E a Frobenius group of order 21 acting faithfully on P, and hence, by Brauer's First Main Theorem, the principal block b_0 of kG is the unique block having P as defect group. By Ward's explicit calculations in [75] or Landrock's general results in [48, §3] we have $|\mathrm{Irr}_K(G,b_0)| = 8$. □

Finally for the triality D_4-groups we have the following proposition due to Deriziotis and Michler [29, Proposition 5.3]

Proposition 15.3. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is simple of type ${}^3D_4(q)$. Let P be a defect group of some block of kG. Then either P is non-abelian or P has rank at most 2. In particular, no block of kG has defect groups which are elementary abelian of order 8.*

16. Unipotent characters with small 2-defects

Recall that the 2-defect of an irreducible character χ of a finite group G is the largest integer $d(\chi)$ such that $2^{d(\chi)}$ divides the rational integer $\frac{|G|}{\chi(1)}$. The notation of the finite groups of Lie type and the labelling of their unipotent characters due to Lusztig in the following two propositions are from the tables page 75/76 and pages 465–488 in Carter's book [20]. In particular, if X is some simple type we denote by $X(q)$ the finite group of Lie type (that is, the group of fixed points in the algebraic group under some Frobenius endomorphism). We will deviate from the notation of [20] in one respect, i.e., we will denote by ${}^2A_l(q)$, ${}^2D_l(q)$, and ${}^3D_4(q)$ the twisted groups denoted by ${}^2A_l(q^2)$, ${}^2D_l(q^2)$, and ${}^3D_4(q^3)$ respectively in [20]. For classical groups, we will also draw upon Olsson's treatment of the combinatorics of symbols [63]. Given a positive integer n, let n_+ denote as before the 2-part of n and n_- the $2'$-part of n. So, $n = n_+n_-$, no odd prime divides n_+ and 2 does not divide n_-.

Proposition 16.1. *Let q be a power of an odd prime r. Let $(q-1)_+ = 2^d$ and $(q+1)_+ = 2^e$.*
(i) Every unipotent character of $A_l(q)$ has 2-defect greater than or equal to dl. If $l+1$ is not a triangular number, then the 2-defect of any unipotent character of $A_l(q)$ is at least $dl + e \geq 2 + l$. If $l \geq 3$, then all unipotent charecters have 2-defect greater than or equal to 5. $A_2(q)$ has one unipotent character, $\chi^{(1,2)}$ of 2-defect $2d$, all other unipotent characters have 2-defect $2d + e \geq 4$. The 2-defect of any unipotent character of $A_1(q)$ is $d + e \geq 3$.
(ii) Every unipotent character of ${}^2A_l(q)$ has 2-defect greater than or equal to el. If $l+1$ is not a triangular number, then the 2-defect of any unipotent character of ${}^2A_l(q)$ is at least $el + d \geq 2 + l$.

If $l \geq 3$, then all unipotent characters have 2-defect greater than or equal to 6. ${}^2A_2(q)$ has one unipotent character, $\chi^{(1,2)}$ of 2-defect $2e$, all other uniptent characters have 2-defect $2e + d \geq 4$. The 2-defect of any unipotent character of ${}^2A_1(q)$ is $d + e \geq 3$. (iii) Every unipotent character of $B_l(q)$ or $C_l(q)$, $l \geq 2$ has 2-defect at least $2l \geq 4$.
(iv) Every unipotent character of $D_l(q)$ or ${}^2D_l(q)$, $l \geq 4$ has 2-defect at least $2l - 1 \geq 7$.

Proof. We note that $d + e \geq 3$.

(i) The unipotent characters of $A_l(q)$ are parametrized by the partitions of $l + 1$. If α is a partition of l and χ^α is the corresponding unipotent character, then

$$|\chi^\alpha(1)|_{r'} = \frac{(q-1)|A_l(q)|_{r'}}{\prod_h (q^h - 1)}$$

where h runs over the set of hook lengths of α (see for instance [54, pp.152-153]). Thus the 2-defect of χ^α is f where $2^f = \frac{\prod_h (q^h-1)_+}{(q-1)_+}$. Since α has $l+1$ hooks, the first assertion follows. If $l+1$ is not a triangular number, then at least one hook of α is of even length. So, $(q^2 - 1)$ divides $q^h - 1$ for at least one hook length h of α, whence $f \geq (d+e)+d(l-1)$. Since 5 is not a triangular number, it is immediate from the first two assertions that any unipotent character of $A_l(q)$, $l \geq 4$ has 2-defect at least 6. Any partition of 4 has two hooks of even length, hence $f \geq 2(d+e) \geq 6$. If $\alpha = (1, 1, 1)$ or $\alpha = (3)$, then the hook lengths of α are 1, 2 and 3 and it follows that $f = 2d+e \geq 4$. If $\alpha = (2, 1)$, then the hook lengths of α are $1, 1$ and 3 and the 2-defect is $2d$. Finally, if $\alpha = (1, 1)$ or (2), then the hook lengths are 1 and 2 and the 2-defect is $d + e$.

(ii) The unipotent characters of ${}^2A_l(q)$ are also parametrised by partitions of $l+1$. If χ^α is the character labelled by α, then

$$|\chi^\alpha(1)|_{r'} = \frac{(q+1)|{}^2A_l(q)|_{r'}}{\prod_h ((-q)^h - 1)},$$

where h runs over the set of hook lengths of α (see [53]). The assertions of (ii) follow as in (i).

(iii) and (iv) Let G be one of the groups $B_l(q)$, $C_l(q)$, $l \geq 2$, $D_l(q)$, ${}^2D_l(q)$, $l \geq 4$. In Lusztig's description, the unipotent characters are labelled using symbols. A symbol is an equivalence class $[X, Y]$ of unordered pairs $[X, Y]$ with X, Y (possibly empty) subsets of the non-negative integers. The pair $[X, Y]$ is equivalent to $[X', Y']$ if and only if there exists an integer t such that either $X = X'^{+t}$ and $Y = Y'^{+t}$ or $X = Y'^{+t}$ and $Y = X'^{+t}$. Let $[X, Y]$ be a symbol, $X = \{a_1, a_2, \cdots, a_k\}$, $Y = \{b_1, b_2, \cdots, b_r\}$, with $a_1 > a_2 \cdots > a_k \geq 0$ and $b_1 > b_2 > \cdots b_r \geq 0$. The rank of $[X, Y]$ is defined by

$$\mathrm{r}k[X,Y] = \sum_i a_i + \sum_j b_j - \lfloor \left(\frac{k+r-1}{2}\right)^2 \rfloor$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Let $c(X, Y)$ be defined by

$$c(X,Y) = \begin{cases} \lfloor \frac{k+r-1}{2} \rfloor - |X \cap Y| & : \quad X \neq Y \\ 0 & : \quad X = Y. \end{cases}$$

If $\chi_{[X,Y]}$ is the unipotent character of G indexed by the symbol $[X, Y]$, then $\mathrm{r}k[X, Y] = l$ and

$$|\chi_{[X,Y]}(1)|_{r'} = \frac{|G|_{r'}}{2^{c(X,Y)} \prod_h (q^h - 1) \prod_{h'} (q^{h'} + 1)}$$

where h runs over the set of hooks of $[X, Y]$ and h' runs over the set of cohooks of $[X, Y]$ (see [63, Proposition 5]). Let $h^+[X, Y]$ be the number of hooks of $[X, Y]$ and let $h^-[X, Y]$ denote the number of cohooks of $[X,Y]$. Then by the character formula above it suffices to prove that $c(X,Y)+h^+[X,Y]+h^-[X,Y]=2(\mathrm{rk}[X,Y])$ in case $G=B_l(q)$ or $G=C_l(q)$ and that $c(X,Y)+h^+[X,Y]+h^-[X,Y]=2(\mathrm{rk}[X,Y])-\delta$, where $\delta\in\{0,1\}$ in case $G=D_l(q)$ or $G={}^2D_l(q)$. By [63, Equations 15, 16],

$$h^+[X,Y]=\sum_i a_i+\sum_j b_j-\binom{k}{2}-\binom{r}{2} \tag{6}$$

and

$$h^-[X,Y]=\sum_i a_i+\sum_j b_j-kr+|X\cap Y|. \tag{7}$$

From these it is straighforward to check that

$$c(X,Y)+h^+[X,Y]+h^-[X,Y]-2(\mathrm{rk}[X,Y])=0$$

if either $k-r$ is odd or $X=Y$ and

$$c(X,Y)+h^+[X,Y]+h^-[X,Y]-2(\mathrm{rk}[X,Y])=-1$$

if $k-r$ is even and $X\neq Y$. The result follows since if $G=B_l(q)$ or $G=C_l(q)$ then $k-r$ is odd and if $G=D_l(q)$ or $G={}^2D_l(q)$ then $k-r$ is even (see [20, Section 13.8]). □

Proposition 16.2. *Let q be an odd prime power.*

(i) $G_2(q)$ has two unipotent characters of 2-defect 0, (labelled $G_2[\theta]$, $G_2[\theta^2]$); if $q\equiv 1 \pmod 4$ then $G_2(q)$ has two unipotent characters of 2-defect 3 (labelled $G_2[1]$, $G_2[-1]$), and if $q\equiv 3 \pmod 4$ then $G_2(q)$ has one unipotent character of 2-defect 3 (labelled $\Phi_{2,2}$). In both cases, all remaining unipotent characters of $G_2(q)$ have 2-defect at least 4.

(ii) ${}^2G_2(q)$ has two cuspidal unipotent characters of 2-defect 0, four cuspidal unipotent characters of 2-defect 2, and the remaining two unipotent characters have 2-defect at least 4.

(iii) $F_4(q)$ has two unipotent characters of 2-defect 0 (labelled $F_4[\theta]$, $F_4[\theta^2]$), and two unipotent characters of 2-defect 5 (labelled $F_4[i]$, $F_4[-i]$). All other unipotent characters of $F_4(q)$ have 2-defect at least 7.

(iv) $E_6(q)$ and ${}^2E_6(q)$ each have two unipotent characters of 2-defect 0 (labelled $E_6[\theta]$, $E_6[\theta^2]$ in the case of $E_6(q)$), and all other unipotent characters have 2-defect at least 8. The unipotent characters of defect 0 of $E_6(q)$ each have degree $\frac{1}{3}q^7\Phi_1^6(q)\Phi_2^4(q)\Phi_4^2(q)\Phi_5(q)\Phi_8(q)$ and the unipotent characters of defect 0 of ${}^2E_6(q)$ each have degree $\frac{1}{3}q^7\Phi_1^4(q)\Phi_2^6(q)\Phi_4^2(q)\Phi_8(q)\Phi_{10}(q)$.

(v) $E_7(q)$ has four unipotent characters of 2-defect $d\geq 3$, labelled $(E_6[\theta],1)$, $(E_6[\theta^2],1)$, $(E_6[\theta],\epsilon)$, $(E_6[\theta^2],\epsilon)$, where 2^d is the largest power of 2 dividing q^2-1. These characters each have degree $\frac{1}{3}q^7\Phi_1^6(q)\Phi_2^6(q)\Phi_4^2(q)\Phi_5(q)\Phi_7(q)\Phi_8(q)\Phi_{10}(q)\Phi_{14}(q)$. Furthermore, $E_7(q)$ has exactly two unipotent characters of 2-defect 8 (labelled $E_7[\xi]$, $E_7[-\xi]$ if $q\equiv 1 \bmod 4$ and $\Phi_{512,11}$, $\Phi_{512,12}$ if $q\equiv 3 \bmod 4$); all other unipotent characters have 2-defect at least 14.

(vi) $E_8(q)$ has four unipotent characters of 2-defect 0 (labelled $E_8[\zeta]$, $E_8[\zeta^2]$, $E_8[\zeta^3]$, $E_8[\zeta^4]$), four unipotent characters of 2-defect 3 (labelled $E_8[\theta]$, $E_8[\theta^2]$, $E_8[-\theta]$, $E_8[-\theta^2]$ if $q\equiv 1 \bmod 4$, and $(E_8[\theta],\Phi_{2,1})$, $(E_8[\theta^2],\Phi_{2,1})$, $(E_8[\theta],\Phi_{2,2}')$, $(E_8[\theta^2],\Phi_{2,2}')$ if $q\equiv 3 \bmod 4$); all other unipotent characters have 2-defect at least 5.

(vii) ${}^3D_4(q)$ has two unipotent characters with 2-defect 3, two unipotent characters with 2-defect at least 5, and four unipotent characters with 2-defect at least 6.

Proof. One proves this by first expressing the group orders given in [20, pp. 75-76] as products of cyclotomic polynomials evaluated at q. Then, for any of the above groups G and any unipotent character Φ of G one calculates the 2-part of $\frac{|G|}{\Phi(1)}$ by running through Lusztig's lists of character degrees of unipotent characters in [20, pp. 477–488], observing that the only cyclotomic polynomials Φ_d which occur in these character degrees for which $\Phi_d(q)$ is even, are those for $d \in \{1, 2, 4, 8\}$. While the 2-part of $\Phi_4(q)$ and $\Phi_8(q)$ is exactly 2, the 2-part of $\Phi_1(q)\Phi_2(q) = q^2 - 1$ is at least 8, possibly bigger, which accounts for the inequalities in the above statements. □

17. Type F_4

Proposition 17.1. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is a simple group of type $F_4(q)$, where q is an odd prime power. If kG has a block b with an elementary abelian defect group P of order 8, then either $|\mathrm{Irr}_K(G, b)| = 8$ or $\mathcal{O}Gb$ is Morita equivalent to a block of $\mathcal{O}L$ of a finite group L such that $|L/Z(L)| < |G/Z(G)|$.*

Proof. Here $Z(G) = 1$ and $G = \mathbf{G}^F$, where $\mathbf{G}$ is a simple algebraic group of type F_4 over $\bar{\mathbb{F}}_q$ and $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $[t]$ be the semi-simple label of b (see §10) and let $\mathbf{C}(t) := C_{\mathbf{G}^*}(t)$. Suppose first that $\mathbf{C}(t)$ is contained in a proper Levi subgroup of $\mathbf{G}^*$. Then by the Jordan decomposition of blocks (see §10), there exists a proper F-stable Levi subgroup, $\mathbf{L}$ such that the block $\mathcal{O}Gb$ is Morita equivalent to some block of $\mathcal{O}\mathbf{L}^F$. Since $\mathbf{L}$ is a proper Levi subgroup of $\mathbf{G}$, $|\mathbf{L}(t)^F/Z(\mathbf{L}(t)^F)|$ is strictly smaller than $|G/Z(G)|$. Hence, we may assume that t is a quasi-isolated element of $\mathbf{G}^*$, i.e. that $\mathbf{C}(t)$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$. Since $Z(\mathbf{G})$ is trivial, $\mathbf{C}(t) = \mathbf{C}^\circ(t)$, hence t is even isolated in $\mathbf{G}^*$, i.e $\mathbf{C}^\circ(t)$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$. From the tables describing centralisers of semi-simple elements in groups of type F_4 in [72], (see also the tables in [28]) one sees that $\bar{\mathbf{C}}^\circ(t)^{F^*}$ is isomorphic to one of $F_4(q)$, $B_4(q)$, $C_3(q) \times A_1(q)$, $A_3(q) \times A_1(q)$, ${}^2A_3(q) \times A_1(q)$, $A_2(q) \times A_2(q)$ or ${}^2A_2(q) \times {}^2A_2(q)$ (and $Z^\circ(\mathbf{C}^\circ(t)) = 1$). By (1), (4) and Propositions 16.1 and 16.2, the 2-defect of any element of $\mathcal{E}(\mathbf{G}^F, [t]) = \mathcal{E}(\mathbf{G}^F, (t))$ is at least 4, whence $\mathcal{E}(\mathbf{G}^F, (t)) \cap \mathrm{Irr}_K(G, b) = \emptyset$, a contradiction. □

18. On characters of small defect in type E groups

We use the notation of §10.

Proposition 18.1. *Suppose that $\mathbf{G}$ is simple, simply connected of type E_6, and $\mathbf{G}^F$ is a group of type $E_6(q)$. Let s be a semi-simple element of $\mathbf{G}^{*F^*}$ and let $\chi \in \mathrm{Irr}_K(G) \cap \mathcal{E}(\mathbf{G}^F, [s])$. Let λ, λ' and $\bar{\lambda}$ correspond to χ as in §10 (3). Suppose that the 2-defect $d(\chi)$ of χ is at most 3. Then a_χ is 1 or 3, $\alpha_\chi = 0$ (see notation after (4) in §10) and s, χ and $\bar{\lambda}$ satisfy one of the rows in the following table.*

	Δ_s	$\mathbf{C}^\circ(s)'^{F^*}$	$z(s)$	$\bar{\lambda}$	$a_\chi\chi(1)_{r'}$	$d(\chi)$	cond. on q
(i)	-	-	$\Phi_1^2\Phi_3^2$	1	$\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	2	$q \equiv 3 \bmod 4$
(ii)	-	-	$\Phi_1^2\Phi_5$	1	$\Phi_1^4\Phi_2^4\Phi_3^3\Phi_4^2\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	2	$q \equiv 3 \bmod 4$
(iii)	-	-	$\Phi_1\Phi_2\Phi_3^2$	1	$\Phi_1^5\Phi_2^3\Phi_3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	3	$(q^2-1)_+ = 8$
(iv)	-	-	$\Phi_1\Phi_2\Phi_5$	1	$\Phi_1^5\Phi_2^3\Phi_3^3\Phi_4^2\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	3	$(q^2-1)_+ = 8$
(v)	-	-	$\Phi_1\Phi_2\Phi_3\Phi_6$	1	$\Phi_1^5\Phi_2^3\Phi_3^2\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi_9\Phi_{12}$	3	$(q^2-1)_+ = 8$
(vi)	-	-	Φ_3^3	1	$\Phi_1^6\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	0	
(vii)	-	-	$\Phi_2^2\Phi_3\Phi_6$	1	$\Phi_1^6\Phi_2^2\Phi_3^2\Phi_4^2\Phi_5\Phi_6\Phi_8\Phi_9\Phi_{12}$	2	$q \equiv 1 \bmod 4$
(viii)	-	-	$\Phi_3\Phi_{12}$	1	$\Phi_1^6\Phi_2^4\Phi_3^2\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9$	0	
(ix)	-	-	Φ_9	1	$\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_{12}$	0	
(x)	-	-	$\Phi_3\Phi_6^2$	1	$\Phi_1^6\Phi_2^4\Phi_3^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}$	0	
(xi)	A_2	$A_2(q)$	Φ_3^2	$\chi^{(2,1)}$	$\Phi_1^4\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$	2	$q \equiv 3 \bmod 4$
(xii)	A_2	${}^2A_2(q)$	$\Phi_3\Phi_6$	$\chi^{(2,1)}$	$\Phi_1^6\Phi_2^2\Phi_3\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}$	2	$q \equiv 1 \bmod 4$
(xiii)	D_4	${}^3D_4(q)$	Φ_3	$\phi_{2,2}$	$\frac{1}{2}\Phi_1^4\Phi_2^4\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}$	3	$q \equiv 3 \bmod 4$
(xiv)	D_4	${}^3D_4(q)$	Φ_3	$\phi_{2,1}$	$\frac{1}{2}\Phi_1^4\Phi_2^4\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9$	3	$q \equiv 3 \bmod 4$
(xv)	D_4	${}^3D_4(q)$	Φ_3	${}^3D_4[-1]$	$\frac{1}{2}\Phi_1^6\Phi_2^2\Phi_3^2\Phi_4^2\Phi_5\Phi_8\Phi_9$	3	$q \equiv 1 \bmod 4$
(xvi)	D_4	${}^3D_4(q)$	Φ_3	${}^3D_4[1]$	$\frac{1}{2}\Phi_1^6\Phi_2^2\Phi_4^2\Phi_5\Phi_8\Phi_9\Phi_{12}$	3	$q \equiv 1 \bmod 4$
(xvii)	$3A_2$	$A_2(q^3)$	1	$\chi^{(2,1)}$	$\Phi_1^4\Phi_2^4\Phi_3\Phi_4^2\Phi_5\Phi_8\Phi_{12}$	2	$q \equiv 3 \bmod 4$
(xviii)	E_6	$E_6(q)$	1	$E_6[\theta]$, $E_6[\theta^2]$	$\frac{1}{3}\Phi_1^6\Phi_2^4\Phi_4^2\Phi_5\Phi_8$	0	

Here $\Phi_d = \Phi_d(q)$ denotes the d-th cyclotomic polynomial over $\mathbb{Q}$ evaluated at q. The non-blank entries of the third column are to be interpreted as $\mathbf{C}^\circ(s)'^{F^*}$ being a group of the given type, and do not specify the isomorphism type of $\mathbf{C}^\circ(s)'^{F^*}$.

Proof. Let s, χ, λ be as in the statement. The group $\mathbf{C}(s)/\mathbf{C}^\circ(s)$ is isomorphic to a subgroup of $\mathrm{Irr}_{\bar{\mathbb{Q}}_{2'}}(Z(\mathbf{G})/Z^\circ(\mathbf{G}))$ (see for instance [26, Lemma 13.14(iii), Remark 2.4]). The first two assertions follow since $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ is a cyclic group of order 3.

We will use (4), and Propositions 16.1 and 16.2 in conjunction with the tables giving connected centralisers of semi-simple elements in groups of type $E_6(q)$ in [28] in order to identify the Δ_s, $\mathbf{C}^\circ(s)^{F^*}$, $z(s)$ and $\bar{\lambda}$ entries of a row in the table. Once these have been identified, the $\frac{\chi(1)_{r'}}{\alpha_\chi}$ entry of the row is calculated using (3), the degree formulae for unipotent characters as given in [20, §13.8] and the order formulae for the finite groups of Lie type (see [20, pp.75-76]). The last two entries are obtained by comparing $\frac{\chi(1)_{r'}}{\alpha_\chi}$ with $|\mathbf{G}^F|$. We note here that

$$|\mathbf{G}^F| = q^{36}\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}.$$

The connected components of Δ_s are of type A_l, $l \geq 1$, D_4, D_5, or E_6. If Δ_s has a component of type E_6 (and in particular is a single component), then s is central in $\mathbf{G}^*$, $z(s) = 1$ and $\bar{\lambda}$ is one of the unipotent characters $E_6[\theta]$ or $E_6[\theta^2]$ of $E_6(q)$ as in Proposition 16.2. Both $E_6[\theta]$ and $E_6[\theta^2]$ have degree $\frac{1}{3}q^7\Phi_1^6\Phi_2^4\Phi_4^2\Phi_5\Phi_8$, hence χ, s, and $\bar{\lambda}$ are as in the last row of the table.

Assume from now on that the components of Δ_s are of classical type. So $\bar{\mathbf{C}}^\circ(s)^{F^*}$ is a direct product of groups $A_l(q^j)$, $l \geq 1$, ${}^2A_l(q^j)$, $l \geq 2$, $D_l(q^j)$, ${}^2D_l(q^j)$, $l \geq 4$ or ${}^3D_4(q)$. By Propositions 16.1 and 16.2, the 2-defect of any unipotent character of a group $A_l(q^j)$, $l \geq 1$, ${}^2A_l(q^j)$, $l \geq 2$, $D_l(q^j)$, ${}^2D_l(q^j)$, $l \geq 4$ or ${}^3D_4(q)$ is at least 2. On the other hand, by (4), $\bar{\lambda}$ has 2-defect at most 3 (note that $\alpha_\chi = 0$). Thus, the connected components of $\Delta(s)$ form one orbit under the action of F^*, and in particular are all pairwise isomorphic. It also follows from Proposition 16.1 that the connected components are not of type D_5 or of type A_l, $l \geq 3$.

Let j be the number of connected components of Δ_s. We claim that the connected components of Δ_s are not of type A_1. Indeed, assume otherwise, so $\mathbf{C}^\circ(s)'^{F^*} = A_1(q^j)$. By Proposition 16.1(i)-(ii), every unipotent character of $A_1(q^j)$ or ${}^2A_1(q^j)$ has 2-defect at least 3. Thus, by (4), $\zeta_s = 0$. But by [28], whenever Δ_s is of type jA_1, $\zeta_s \geq 1$, proving the claim.

Suppose that the connected components of Δ_s are of type A_2, $1 \leq j \leq 3$. If $j = 3$, then by [28], $\mathbf{C}^\circ(s)'^{F^*} = A_2(q^3)$, $z(s) = 1$, and by Proposition 16.1(i), $\bar{\lambda}$ is the character $\chi^{(2,1)}$. By the hook-length degree formula for unipotent characters of groups of type A (see the proof of Proposition 16.1(i)),

$$\bar{\lambda}(1)_{r'} = \frac{(q^3-1)|A_2(q^3)|}{(q^9-1)(q^3-1)^2} = \frac{|A_2(q^3)|}{\Phi_1^2\Phi_3^2\Phi_9},$$

we see that s and χ are as in row (xvii) of the table. If $j = 2$, then by [28], $\mathbf{C}^\circ(s)'^{F^*}(s) = A_2(q^2)$, whereas by Proposition 16.1 any unipotent character of $A_2(q^2)$ has 2-defect at least 6 (note that $(q^2-1)_+ \geq 8$). Thus this case does not occur.

If $j = 1$, and $\mathbf{C}^\circ(s)'^{F^*} = A_2(q)$, then $\zeta_s \leq 1$, hence by [28], $z(s)$ is (q^2+q+1) and s and χ are as in row (xi) of the table. If $j = 1$, and $\mathbf{C}^\circ(s)'^{F^*} = {}^2A_2(q)$, then $\zeta_s \leq 1$, hence by [28], $z(s)$ is (q^4+q^2+1) and s and χ are as in row (xii) of the table.

Suppose that the connected components of Δ_s are of type D_4, so $j = 1$. By Proposition 16.1(iv), and by [28], $\mathbf{C}^\circ(s)'^{F^*} = {}^3D_4(q)$ and $z(s) = q^2+q+1$. By Proposition 16.2(vii) $\bar{\lambda}$ is one of $\phi_{2,2}$ or $\phi_{2,1}$ if $q \equiv 3 \pmod 4$ and $\bar{\lambda}$ is one of ${}^3D_4[-1]$ or ${}^3D_4[1]$ if $q \equiv 1 \pmod 4$. Hence, χ, s and λ are as in lines (xiii)-(xvii) of the table.

Finally assume that $\mathbf{C}^\circ(s)$ is a torus, so $\mathbf{C}^\circ(s)'$ and $\bar{\lambda}$ are trivial. By (4) we have $z(s) = d(\chi) \leq 3$. An inspection of [28] yields that $z(s)$ is as in the first ten rows of the table. $\square$

Proposition 18.2. *Suppose that $\mathbf{G}$ is simple, simply connected of type E_6, and $\mathbf{G}^F$ is a group of type ${}^2E_6(q)$. Let s be a semi-simple element of $\mathbf{G}^{*F^*}$ and let $\chi \in \mathrm{Irr}_K(G) \cap \mathcal{E}(\mathbf{G}^F, [s])$. Let $\bar{\lambda}$ correspond to χ as in (3). Suppose that the 2-defect $d(\chi)$ of χ is at most 3. Then a_χ is 1 or 3, $\alpha_\chi = 0$ and s, χ and $\bar{\lambda}$ satisfy one of the rows in the following table.*

	Δ_s	$\mathbf{C}^\circ(s)'^{F^*}$	$z(s)$	$\bar{\lambda}$	$a_\chi\chi(1)_{r'}$	$d(\chi)$	cond. on q
(i)	-	-	$\Phi_2^2\Phi_6^2$	1	$\Phi_2^4\Phi_1^4\Phi_6\Phi_4^2\Phi_{10}\Phi_3^2\Phi_8\Phi_{18}\Phi_{12}$	2	$q \equiv 1 \bmod 4$
(ii)	-	-	$\Phi_2^2\Phi_{10}$	1	$\Phi_2^4\Phi_1^4\Phi_6^3\Phi_4^2\Phi_3^2\Phi_8\Phi_{18}\Phi_{12}$	2	$q \equiv 1 \bmod 4$
(iii)	-	-	$\Phi_2\Phi_1\Phi_6^2$	1	$\Phi_2^5\Phi_1^3\Phi_6\Phi_4^2\Phi_{10}\Phi_3^2\Phi_8\Phi_9\Phi_{12}$	3	$(q^2-1)_+ = 8$
(iv)	-	-	$\Phi_2\Phi_1\Phi_{10}$	1	$\Phi_2^5\Phi_1^3\Phi_6^3\Phi_4^2\Phi_3^2\Phi_8\Phi_{18}\Phi_{12}$	3	$(q^2-1)_+ = 8$
(v)	-	-	$\Phi_2\Phi_1\Phi_6\Phi_3$	1	$\Phi_2^5\Phi_1^3\Phi_6^2\Phi_4^2\Phi_{10}\Phi_3\Phi_8\Phi_{18}\Phi_{12}$	3	$(q^2-1)_+ = 8$
(vi)	-	-	Φ_6^3	1	$\Phi_2^6\Phi_1^4\Phi_4^2\Phi_{10}\Phi_9^2\Phi_8\Phi_{18}\Phi_{12}$	0	
(vii)	-	-	$\Phi_1^2\Phi_6\Phi_3$	1	$\Phi_2^6\Phi_1^2\Phi_6^2\Phi_4^2\Phi_{10}\Phi_3\Phi_8\Phi_{18}\Phi_{12}$	2	$q \equiv 3 \bmod 4$
(viii)	-	-	$\Phi_6\Phi_{12}$	1	$\Phi_2^6\Phi_1^4\Phi_6^2\Phi_4^2\Phi_{10}\Phi_3^2\Phi_8\Phi_{18}$	0	
(ix)	-	-	Φ_{18}	1	$\Phi_2^6\Phi_1^4\Phi_6^3\Phi_4^2\Phi_{10}\Phi_3^2\Phi_8\Phi_{12}$	0	
(x)	-	-	$\Phi_6\Phi_3^2$	1	$\Phi_2^6\Phi_1^4\Phi_6^2\Phi_4^2\Phi_{10}\Phi_8\Phi_{18}\Phi_{12}$	0	
(xi)	A_2	${}^2A_2(q)$	Φ_6^2	$\chi^{(2,1)}$	$\Phi_2^4\Phi_1^4\Phi_4^2\Phi_{10}\Phi_3^2\Phi_8\Phi_{18}\Phi_{12}$	2	$q \equiv 1 \bmod 4$
(xii)	A_2	$A_2(q)$	$\Phi_6\Phi_3$	$\chi^{(2,1)}$	$\Phi_2^6\Phi_1^2\Phi_6\Phi_4^2\Phi_{10}\Phi_8\Phi_{18}\Phi_{12}$	2	$q \equiv 3 \bmod 4$
(xiii)	D_4	${}^3D_4(q)$	Φ_6	${}^3D_4[1]$	$\frac{1}{2}\Phi_2^4\Phi_1^4\Phi_4^2\Phi_{10}\Phi_8\Phi_{18}\Phi_{12}$	3	$q \equiv 1 \bmod 4$
(xiv)	D_4	${}^3D_4(q)$	Φ_6	${}^3D_4[-1]$	$\frac{1}{2}\Phi_2^4\Phi_1^4\Phi_4^2\Phi_{10}\Phi_3^2\Phi_8\Phi_{18}$	3	$q \equiv 1 \bmod 4$
(xv)	D_4	${}^3D_4(q)$	Φ_6	$\phi_{2,1}$	$\frac{1}{2}\Phi_2^6\Phi_1^2\Phi_6^2\Phi_4^2\Phi_{10}\Phi_8\Phi_{18}$	3	$q \equiv 3 \bmod 4$
(xvi)	D_4	${}^3D_4(q)$	Φ_6	$\phi_{2,2}$	$\frac{1}{2}\Phi_2^6\Phi_1^2\Phi_4^2\Phi_{10}\Phi_8\Phi_{18}\Phi_{12}$	3	$q \equiv 3 \bmod 4$
(xvii)	$3A_2$	${}^2A_2(q^3)$	1	$\chi^{(2,1)}$	$\Phi_2^4\Phi_1^4\Phi_6\Phi_4^2\Phi_{10}\Phi_8\Phi_{12}$	2	$q \equiv 1 \bmod 4$
(xviii)	E_6	${}^2E_6(q)$	1	${}^2E_6[\theta]$, ${}^2E_6[\theta^2]$	$\frac{1}{3}\Phi_2^6\Phi_1^4\Phi_4^2\Phi_{10}\Phi_8$	0	

Proof. This is exactly as in the proof of Proposition 18.1, replacing q by $-q$ and noting that this switches the roles of the pairs Φ_1 and Φ_2, Φ_3 and Φ_6, Φ_5 and Φ_{10}, and Φ_9 and Φ_{18}. □

19. Type E in odd characteristic

Let q be an odd prime power. We do not know whether there are any blocks with elementary abelian defect group of order 8 in type E over a field of odd characteristic - the next result says that *if* there is such a block, it is either nilpotent or its inertial quotient has order 7.

Proposition 19.1. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is a simple group of type $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$. If kG has a block b with an elementary abelian defect group P of order 8, then the inertial quotient of b is either trivial (in which case b is nilpotent) or has order 7.*

Proof. It suffices to show that if $u \in P$ is an involution and if e is a block of $kC_G(u)$ with defect group P then e is nilpotent. Indeed, there are non-nilpotent (G, b)-Brauer elements if and only if the inertial quotient of b has order 3 or 21. The pattern of the proof is this: using the lists of centralisers in [39, 4.5.1, 4.5.2], we show that in 'most' cases, $C_G(u)$ has a direct factor of the form $2.H.2$ for some finite group H. As in [24], this means that $2.H$ is a central extension of H by an involution such that $2.H$ is contained as a subgroup of index 2 in $2.H.2$, and an automorphism α of $2.H$ induced by conjugation with a 2-element a in $2.H.2 - 2.H$ has an image $\bar{\alpha}$ in the outer automorphism group of $2.H$ of order 2. Using [24, 6.4] and again [39, 4.5.1, 4.5.2], we will then show that α stabilises all blocks of H, implying that a is contained in a defect group of any block, and so any block of $2.H.2$ with an elementary abelian defect group P is necessarily nilpotent, because its image in $H.2$ is a block with a Klein four defect group which is also a 2-extension of a block of H with a defect group of order 2. Here are the details for the different groups; this follows arguments in [24, §12]. All group theoretic facts about centralisers of involutions are from [39, §4].

Suppose that $G/Z(G)$ is of type $E_6(q)$. There are two classes of involutions in G, with representatives t_1, t_2. If $u = t_1$ then $C_G(u)$ has a central cyclic subgroup of order $\gcd(4, q-1)$, hence $q \equiv 3 \pmod 4$ since P has exponent 2. Then $C_G(u) \cong 2.\,\mathrm{P\Omega}_{10}(q).2 \times C_{(q-1)/2}$. The automorphism α of $\mathrm{P\Omega}_{10}$ as constructed above is inner diagonal, hence [24, 6.4] implies that it is contained in a defect group of any block of $C_G(u)$. Thus any block of $C_G(u)$ with P as defect group is nilpotent by the argument outlined at the beginning of the proof. If $u = t_2$ then $C_G(u) \cong 2.(\mathrm{PSL}_2(q) \times Z(N).\,\mathrm{PSL}_6(q)).2$. Again, α is inner diagonal on both factors, and so the same argument using [24, 6.4] shows that any block of $C_G(u)$ with defect group P is nilpotent.

Suppose that $G/Z(G)$ is of type ${}^2E_6(q)$. There are two classes of involutions in G, with representatives t_1, t_2. If $u = t_1$ then $C_G(u)$ has a central cyclic subgroup of order $\gcd(4, q+1)$, hence $q \equiv 1 \pmod 4$ since P has exponent 2. Then $C_G(u) \cong 2.\,\mathrm{P\Omega}^-_{10}(q).2 \times C_{(q+1)/2}$. If $u = t_2$ then $C_G(u) \cong 2.(\mathrm{PSL}_2(q) \times (3, q+1).\,\mathrm{PSU}_6(q)).2$, and the same argument as for $E_6(q)$ shows that in both cases, any block of $kC_G(u)$ with defect group P is nilpotent.

Suppose that $G/Z(G)$ is of type $E_7(q)$. In that case $Z(G) = \{1\}$, so G is simple, but the finite group of Lie type $E_7(q)$ is a central extension of G by an involution. Let $E = \mathrm{Inndiag}(G)$; then $|E : G| = 2$. Table 4.5.1 in [39] describes $C_E(u)$, for any involution u in G, and involutions in G which are E-conjugate will have isomorphic centralisers in G. There are five E-conjugacy classes of involutions in G, represented by t_1, t_4, t'_4, t_7 and t'_7. If $u = t_1$ then $C_E(u) \cong 2.(\mathrm{PSL}_2(q) \times \mathrm{P\Omega}_{12}(q)).(C_2 \times C_2)$, so our standard argument implies that any block of $C_G(u)$ with P as defect group is nilpotent. If $u = t_4$ then $q \equiv 5 \pmod 8$ (else t_4 is non-inner), and $C_E(u) \cong$

$2.\,\mathrm{PSL}_8(q).C_{\gcd(8,q-1)}.\gamma$, where γ is a graph automorphism. It follows from considering the table 4.5.2 in [39] applied to the inverse image of u in $E_7(q)$ that γ is not in G, hence $C_G(u) \cong 2.\,\mathrm{PSL}_8(q).C_{\gcd(8,q-1)}$. Since $\gcd(8, q-1)$ is at least 2, we get again that any block of $C_G(u)$ with P as defect group is nilpotent. If $u = t'_4$ then $q \equiv 3 \pmod 4$ (or else t'_4 is non-inner) and $C_E(u) \cong 2.\,\mathrm{PSU}_8(q).C_{\gcd(8,q-1)}.\gamma$. As in the previous case we get $C_G(u) \cong 2.\,\mathrm{PSU}_8(q).C_{\gcd(8,q-1)}$, and thus that any block of $C_G(u)$ with P as defect group is nilpotent. If $u = t_7$ then $q \equiv 1 \pmod 4$ and $C_E(q) \cong (\gcd(3, q-1).E_6(q).3 * C_{q-1}).2$, and hence, by an argument as before (using [39, 4.5.2]), we get that $C_G(u) \cong (\gcd(3, q-1).E_6(q).3 * C_{q-1})$. This shows that $C_G(u)$ has a normal cyclic subgroup of order 4, hence has no block with P as defect group. Similarly, if $u = t'_7$ then $q \equiv 3 \pmod 4$ and $C_E(q) \cong (\gcd(3, q+1).{}^2E_6(q).3 * C_{q+1}).2$; as above we get $C_G(u) \cong (\gcd(3, q+1).{}^2E_6(q).3 * C_{q+1})$. This shows that $C_G(u)$ has again a normal cyclic subgroup of order 4, hence has no block with P as defect group.

Suppose finally that $G/Z(G)$ is of type $E_8(q)$. Then $Z(G) = \{1\}$ and G has two conjugacy classes of involutions, with representatives t_1 and t_8. Their centralisers are isomorphic to $2.\,\mathrm{P}\Omega_{16}(q).2$ and $2.(\mathrm{PSL}_2(q) \times \bar{E}_7(q)).2$, where $\bar{E}_7(q)$ is the simple quotient of $E_7(q)$. In both cases our standard argument shows that every block with defect group P is nilpotent. □

Notation. For each $n > 2$, let p_n be a Zsigmondy prime for the pair (q, n), i.e., p_n is a prime number such that p_n divides $q^n - 1$ but p_n does not divide $q^m - 1$ for any $m < n$. Such p_n exist by Zsigmondy's theorem (see [69, Theorem 3]), and note that q is odd and that n is assumed greater than 2. Since q has order n modulo p_n, if p_n divides $\Phi_d(q)$ for some natural number d then p_n divides $q^d - 1$, whence $n|d$.

Proposition 19.2. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is a simple group of type $E_6(q)$. If kG has a block b with an elementary abelian defect group P of order 8, then either $|\mathrm{Irr}_K(G, b)| = 8$ or $\mathcal{O}Gb$ is Morita equivalent to a block of $\mathcal{O}L$ of a finite group L such that $|L/Z(L)| < |G/Z(G)|$.*

Proof. The centre $Z(G)$ has order 1 or 3 and we may assume without loss of generality that $Z(G)$ has order 3. Hence $G = \mathbf{G}^F$, where $\mathbf{G}$ is a simply connected simple algebraic group of type E_6 over $\bar{\mathbb{F}}_q$ and $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Keeping the notation of the previous section let $[t]$ be the semi-simple label of b. Arguing as for Proposition 17.1, we may and we will assume that t is quasi-isolated. Suppose, if possible that $|\mathrm{Irr}_K(G, b)| \neq 8$. By Propositions 3.3 and 19.1,

$$|\mathrm{Irr}_K(G, b)| = 5 \quad \text{and} \quad \mathrm{Irr}_K(G, b) = \{\chi_j,\ 1 \le j \le 5\}$$

where χ_1 has height one and χ_j has height zero for $2 \le j \le 5$. For each j, $1 \le j \le 5$, let s_j be a semi-simple element in $\mathbf{G}^{*F^*}$ such that $\chi_j \in \mathcal{E}(\mathbf{G}, [s_j])$. By [16, Théorème 2.2], for any $1 \le j \le 5$, s_j can be chosen to have the form $s_j = tu_j$, where u_j is a 2-element of $\mathbf{C}(t)^{F^*}$. Since $\mathbf{C}(t)/\mathbf{C}^\circ(t)$ has exponent dividing the order of t, (see for example [26, Remark 13.15 (i)]), $u_j \in \mathbf{C}^\circ(t)^{F^*}$. In particular, the connected components of Δ_{s_j} are subdiagrams of the extended connected components of Δ_t. Here, by an extended Dynkin diagram we mean a completed Dynkin diagram in the sense of [9, Chapter 6, §4.3].

Since t is quasi-isolated and has odd order, by [7, Table III], Δ_t is one of E_6, $A_2 \times A_2 \times A_2$, or D_4. On the other hand, since $\mathcal{E}(\mathbf{G}, [t]) \cap \mathrm{Irr}_K(G, b) \neq \emptyset$, $\mathcal{E}(\mathbf{G}, [t])$ and hence $\mathcal{E}(\mathbf{G}, (t))$ contains an element of 2-defect 2 or 3. By Equation 4 and Proposition 18.1, it follows that $C_{\mathbf{G}}(t)^{F^*}$ contains a unipotent character of 2-defect at most 3 and at least 2. Now if Δ_t is of type E_6, then t is central in $\mathbf{G}^*$, i.e. $C_{\mathbf{G}}(t)^{F^*} = E_6(q)$. But by Proposition 16.2 the 2-defect of a unipotent character of a group of type $E_6(q)$ is either 0 or at least 8, a contradiction.

Suppose that $\Delta_t = 3A_2$. Then, since $C_{\mathbf{G}}(t)^{\circ F^*}$ contains a unipotent character of 2-defect 2 or 3, we see that t is as in row (xvii) of Table 18.1. But since the characters corresponding to (xvii) have 2-defect 2, it follows by Proposition 3.3 that $u_1 = 1$ and that u_j is non-trivial for all j, $2 \leq j \leq 5$.

Let $2 \leq j \leq 5$. Since the 2-defect of s_j is 3 and since Δ_{s_j} is not of type D_4 or E_6, $C_{\mathbf{G}^*}(s_j)^\circ$ is a torus corresponding to rows (iii), (iv) or (v) of Table 18.1. In particular, $\Phi_9(q)$ and hence p_9 is a divisor of $\chi_j(1)$ for all j, $2 \leq j \leq 5$. On the other hand, χ_1 corresponds to row (xvii) of Table 18.1 and we see that p_9 does not divide $\chi_1(1)$. Hence, p_9 does not divide $2\chi_1(1) - \sum_{2\leq j\leq 5} \delta_j\chi_j(1)$. In particular, $2\chi_1(1) - \sum_{2\leq j\leq 5} \delta_j\chi_j(1) \neq 0$, contradicting Proposition 3.3(ii).

So $\Delta_t = D_4$. Let j_0 be such that $s_{j_0} = t$, $1 \leq j_0 \leq 2$. Then t corresponds to one of rows (xiii)-(xvi) of Table 18.1. Since the characters corresponding to these rows have 2-defect 2, $j_0 \geq 2$, say $j_0 = 2$. Let us first consider the case that χ_2 corresponds to row (xiii) of the table, so $q \equiv 3 \pmod 4$ and let l, $3 \leq l \leq 5$ be such that (s_l, χ_l) also corresponds to row (xiii) of the table. Then semi-simple part of $C_{\mathbf{G}^*}(s_l)^\circ$, and hence of $C_{C_{\mathbf{G}^*(t)^\circ}}(u_l)$ is of type D_4. Thus u_l is central in $C_{\mathbf{G}^*}(t)^\circ$. Since u_l is a 2-element, and ζ_t is odd, we get that u_l is a central element of $[C_{\mathbf{G}^*}(t)^\circ, C_{\mathbf{G}^*}(t)^\circ]$. But the group ${}^3D_4(q)$ has a trivial centre, hence $u_1 = 1$. Thus, if (s_l, χ_l) corresponds to row (xiii) of Table 18.1, then $[s_l] = [t]$. By [7], t is quasi-isolated but not isolated and $C_{\mathbf{G}^*}(t)^\circ$ is of index 3 in $C_{\mathbf{G}^*}(t)$. Since $\phi_{2,1}$ is the unique unipotent character of its degree in ${}^3D_4(q)$, $\phi_{2,1}$ is stable under $C_{\mathbf{G}^{F^*}}(t)$ and hence there are three possibilities for χ_l, each of degree

$$\delta = \frac{1}{6}q^{33}\Phi_1^4(q)\Phi_2^4(q)\Phi_4^2(q)\Phi_5(q)\Phi_8(q)\Phi_9(q)\Phi_{12}(q).$$

Let $\chi_1 - \sum_{2\leq i\ leq5} \delta_i\chi_i$ be the element of $L^0(G,b)$ as in Proposition 3.3. Then, $\delta_l = \delta_2$. Thus, from the degree formula above, it follows that $\sum_i \delta_i\chi_i(1)$, where i ranges over the indices for which s_i corresponds to row (xiii) of Table 18.1 is not divisible by the Zsigmondy prime p_6 (note that the number of such indices is at most 3).

Now let j, $1 \leq j \leq 5$ be such that s_j does not correspond to row (xiii) of Table 18.1. Since $q \equiv 3 \pmod 4$, and Δ_{s_j} is a subdiagram of the extended Dynkin diagram of Δ_t, (s_j, χ_j) corresponds to one of rows (i), (ii), (iii), (iv), (v), (xi), or (xiv) of Table 18.1. In particular, from the character degree column of Table 18.1, we see that $\chi_j(1)$ is divisible by p_6 for any such j. Hence, $2\chi_1(1) - \sum_{2\leq i\leq 5} \delta_i\chi_i(1)$ is not divisible by p_6, a contradiction.

We get a similar contradiction if (t, χ_2) corresponds to row (xvi) of Table 18.1 and with p_6 replaced by p_{12} if (t, χ_2) corresponds to row (xiv) or row (xv) of Table 18.1. □

Proposition 19.3. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is a simple group of type ${}^2E_6(q)$. If kG has a block b with an elementary abelian defect group P of order 8, then either $|\mathrm{Irr}_K(G,b)| = 8$ or $\mathcal{O}Gb$ is Morita equivalent to a block of $\mathcal{O}L$ of a finite group L such that $|L/Z(L)| < |G/Z(G)|$.*

Proof. This is as the proof of Proposition 19.2, with the roles of q and $-q$ suitably reversed. □

Proposition 19.4. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is a simple group of type $E_7(q)$. Suppose that kG has a block b with an elementary abelian defect group P of order 8 such that $|\mathrm{Irr}_K(G,b)| \neq 8$. Then, there exists a finite group L with $|L/Z(L)| < |G/Z(G)|$ and a block c of $\mathcal{O}L$ with elementary abelian defect groups of order 8 and such that $|\mathrm{Irr}_K(L,c)| \neq 8$.*

Proof. First note that $Z(G) = 1$ and that $G = \tilde{G}/\langle z\rangle$, where $\tilde{G} = \tilde{\mathbf{G}}^F$ for a simply-connected simple algebraic group $\tilde{\mathbf{G}}$ of type E_7, $F : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ is a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\tilde{\mathbf{G}}$, and $\langle z\rangle$ is the central subgroup of order 2 of $\tilde{\mathbf{G}}$. Let $\tilde{b}$ be the block of $\mathcal{O}\tilde{G}$ lifting b and let $[t]$ be the semi-simple label of $\tilde{b}$.

Suppose first that t is not a quasi-isolated element of $\tilde{\mathbf{G}}^*$. Then, arguing as for Proposition 17.1, there exists a proper F-stable Levi subgroup $\tilde{\mathbf{L}}$ of $\tilde{\mathbf{G}}$, and a block $\tilde{c}$ of $\mathcal{O}\tilde{\mathbf{L}}^F$ such that $\mathcal{O}\tilde{\mathbf{G}}^F\tilde{b}$ and $\mathcal{O}\tilde{\mathbf{L}}^F\tilde{c}$ are Morita equivalent. Noting that $z \in \tilde{\mathbf{L}}^F$, set $L = \tilde{\mathbf{L}}^F/\langle z\rangle$ and let c be the image in $\mathcal{O}L$ of $\tilde{c}$. By Proposition 3.5, the defect groups of c are elementary abelian and c does not satisfy Alperin's weight conjecture. Since

$$|L/Z(L)| \leq L = \frac{|\tilde{\mathbf{L}}^F|}{2} < \frac{|\tilde{\mathbf{G}}^F|}{2} = |G| = |G/Z(G)|,$$

the result follows.

We assume from now on that t is quasi-isolated in $\tilde{\mathbf{G}}^*$. Before proceeding we note that since t has odd order, $\mathbf{C}_{\tilde{\mathbf{G}}^*}(t)$ is connected and hence t is in fact isolated. By table III of [7], and using that t is of odd order, we get that either $t = 1$ and the Dynkin diagram Δ_t corresponding to $C_{\tilde{\mathbf{G}}}(t)$ is of type E_7 or t has order 3 and Δ_t is of type $A_2 \times A_5$. Since there is an ordinary irreducible character of $\tilde{b}$ in the $\mathcal{E}(\tilde{\mathbf{G}}^F, [t])$, it follows from Equation (3) in §10 that $(\mathbf{C}_{\tilde{\mathbf{G}}^*}(t)/Z(\mathbf{C}_{\tilde{\mathbf{G}}^*}(t)))^{F^*}$ has a unipotent character of defect at most 4 (note that $\tilde{b}$ has defect groups of order 16). By Proposition 16.1 it follows that Δ_t is not of type $A_2 \times A_5$. So, Δ_t is of type E_7, i.e., $\tilde{b}$ is a unipotent block. But the defect groups of all non-principal unipotent blocks of $\tilde{\mathbf{G}}^F$ which are not central in $\tilde{\mathbf{G}}^F$ are dihedral groups ([30, p.357]). On the other hand, a defect group of $\tilde{b}$ is a central extension of an elementary abelian group of order 8 by a group of order 2, a contradiction. □

Proposition 19.5. *Let G be a quasi-simple finite group such that $Z(G)$ has odd order and such that $G/Z(G)$ is a simple group of type $E_8(q)$. Suppose that kG has a block b with an elementary abelian defect group P of order 8 such that $|\mathrm{Irr}_K(G, b)| \neq 8$. Then, there exists a finite group L with $|L/Z(L)| < |G/Z(G)|$ and a block c of $\mathcal{O}L$ with elementary abelian defect groups of order 8 and such that $|\mathrm{Irr}_K(L, c)| \neq 8$.*

Proof. Note that $G = \mathbf{G}^F$ for a simply-connected simple algebraic group $\mathbf{G}$ of type E_8, and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius morphism with respect to an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let $[t]$ be the semi-simple label of b. Again, the Bonnafé-Rouquier result allows us to reduce to the case that t is a quasi-isolated (hence isolated) element of $\mathbf{G}^*$. By the tables in [27], one sees that Δ_t is one of $A_4 \times A_4$, $A_5 \times A_2 \times A_1$, $A_7 \times A_1$, A_8, $D_5 \times A_3$, D_8, $E_6 \times A_2$, $E_7 \times A_1$ or E_8. Using as before that $(C_{\mathbf{G}^*}(t)/Z(C_{\mathbf{G}^*}(t)))^{F^*}$ has a unipotent character of defect at most 3 (see Equation (3) in §10) and Propositions 16.1 and 16.2 we have that Δ_t is one of $E_6 \times A_2$, or E_8. By [30, p.364], the defect groups of any unipotent block of positive defect of G have order at least 16 hence Δ_t is not of type E_8.

Suppose that Δ_t is of type $E_6 \times A_2$ and set $\mathbf{C} = \mathbf{C}_{\mathbf{G}^*}(t)$. Since $\mathbf{C}$ is connected and has centre of odd order, by Propositions 16.1 and 16.2, and the discussion preceding Proposition 18.1, $\mathbf{C}^{F^*}$ has no unipotent character of defect 3, and at most one unipotent character of 2-defect 2 (corresponding to the product of the unipotent character of 2-defect 2 of $A_2(q)$ or ${}^2A_2(q)$ and a unipotent character of 2-defect zero of $E_6(q)$ or ${}^2E_6(q)$). We have $|\mathrm{Irr}_K(G, b)| = 5$ and can write $\mathrm{Irr}_K(G, b) = \{\chi_j,\ 1 \leq j \leq 5\}$ as in Proposition 3.3. For each j, $1 \leq j \leq 5$, let s_j and u_j be as in the proof of Proposition 19.2. Let j_0, $1 \leq j_0 \leq 5$ be such that χ_{j_0} is in the rational series indexed by $[t]$. By the above discussion $j_0 = 1$ and χ_1 is the unique character of b in the rational Lusztig series corresponding to $[t]$. In other words, u_j is a non-trivial 2-element for $2 \leq j \leq 5$.

Let $2 \leq j \leq 5$. Set $\mathbf{C}_j = C_{\mathbf{G}^*}(s_j) = \mathbf{C}^{\circ}_{\mathbf{G}^*}(s_j)$, $\mathbf{Z}_j = Z(C_{\mathbf{G}^*}(s_j))$ and $\bar{\mathbf{C}}_j = \mathbf{C}_j/\mathbf{Z}_j$. Set $z_j = |\mathbf{Z}_j^{F^*}|$ and let ζ_j be defined by $2^{\zeta_j} = z_{j_+}$. Further, let $\bar{\lambda}_j$ be a unipotent character of $\bar{\mathbf{C}}_j^{F^*}$ corresponding to χ_j as in the discussion preceding Equation 4, i.e., such that

$$\chi_{j_0}(1) = \frac{|G|_{r'}}{z_j|\bar{C}_j^{F^*}|}\bar{\lambda}_j(1) \tag{8}$$

and

$$\zeta_j + d(\tau_j) = 3 \tag{9}$$

where $d(\tau_j)$ is the 2-defect of τ_j. We have that $\mathbf{C} := \mathbf{C}_{\mathbf{G}^*}(t) = \mathbf{X}.\mathbf{Y}$ where $\mathbf{X}$ is a simply connected group of type A_2 and $\mathbf{Y}$ is a simple group of type E_6, $\mathbf{X}$ and $\mathbf{Y}$ commute and intersect in a central subgroup of order 3. As u_j is a non-trivial F^*-stable 2-element, there are unique F^*-stable 2-elements $v_1 \in \mathbf{X}$, $v_2 \in \mathbf{Y}$, at least one of which is non-trivial and such that $u_j = v_1 v_2$,

$$C_{\mathbf{G}^*}(s_j) = C_{\mathbf{X}}(v_1)C_{\mathbf{Y}}(v_2)$$

and $C_{\mathbf{G}}(s_j)^{F^*}$ contains $C_{\mathbf{X}}(v_1)^{F^*}C_{\mathbf{Y}}(v_2)^{F^*}$ as a normal subgroup of index 3.

In particular, $\langle v_1, v_2\rangle$ is a central subgroup of $C_{\mathbf{G}^*}(s_j)^{F^*}$. Since the centre of $C_{\mathbf{G}^*}(s_j)^{F^*}$ is contained in the kernel of any unipotent character of $C_{\mathbf{G}^*}(s_j)^{F^*}$ it follows that the product of the orders of v_1 and v_2 is at most 8.

First suppose that both $C_{\mathbf{X}}(v_1)$ and $C_{\mathbf{Y}}(v_2)$ are tori. Then, both v_1 and v_2 are non-trivial. On the other hand, the product of the orders of v_1 and v_2 is at most 8, hence at least one of v_1 and v_2 has order 2. But the centraliser of an element of order 2 in a simple algebraic group of type A_2 or E_6 is not a torus (see [39, Table 4.3.1]), a contradiction.

Suppose that neither $C_{\mathbf{X}}(v_1)$ nor $C_{\mathbf{Y}}(v_2)$ is a torus. We have

$$\bar{\mathbf{C}}_j = C_{\mathbf{X}}(v_1)/Z(C_{\mathbf{X}}(v_1)) \times C_{\mathbf{Y}}(v_2)/Z(C_{\mathbf{Y}}(v_2)).$$

Hence, $\bar{\lambda}_j = \phi_1 \times \phi_2$, where ϕ_1 is a unipotent character of $C_{\mathbf{X}}(v_1)/Z(C_{\mathbf{X}}(v_1))^{F^*}$ and ϕ_2 is a unipotent character of $C_{\mathbf{Y}}(v_2)/Z(C_{\mathbf{Y}}(v_2))^{F^*}$, such that the sum of ζ_j and the 2-defects of ϕ_1 and ϕ_2 equals 3. Since the only subdiagrams of the extended Dynkin diagram of type A are of type A, it follows from [27] and Propositions 16.1 and 16.2 that either Δ_j is $A_1 \times E_6$, and ζ_j has odd order or Δ_j is $A_2 \times E_6$ and ζ_j has even order. But by [27], there are no such s_j.

Thus, exactly one of $C_{\mathbf{X}}(v_1)$ or $C_{\mathbf{Y}}(v_2)$ is a torus. Then $\zeta_j > 0$ (v_i is a 2-element) so by Equation (9), the 2-defect of $\bar{\lambda}_j$ is at most 2. From Propositions 16.1 and 16.2, it follows that either Δ_j is a product of copies of A_2 transitively permuted by F^* or Δ_j is of type E_6.

If Δ_j is a product of copies of A_2 transitively permuted by F^*, then by the tables in [27], one sees that either $\zeta_j = 0$ or $\zeta_j \geq 2$, whence by Equation (9), $\bar{\lambda}_j$ has 2-defect 3 or at most 1. This is a contradiction as any unipotent character of $A_2(q^m)$ or ${}^2A_2(q^m)$ has 2-defect 2 or greater than 3.

If Δ_j is of type E_6, then $\bar{\lambda}_j$ has 2-defect 0, whence the Sylow 2-subgroups of $\mathbf{Z}_j^{F^*}$ have order 8. Since Δ_j is a subdiagram of the extended diagram associated to $A_2 \times E_6$, it follows that $C_{\mathbf{X}}(v_1)$ is a torus and that the Sylow 2-subgroups of $C_{\mathbf{X}}(v_1)^{F^*}$ have order 8. From [27] (or from the description of F^*-stable maximal tori in type A_2), it follows that $|C_{\mathbf{X}}(v_1)^{F^*}|$ is one of (q^2-1), $(q \pm 1)^2$, or $(q^2 \pm q + 1)$. Thus the only possibility is that $|C_{\mathbf{X}}(v_1)^{F^*}| = (q^2-1)$ and 8 is the highest power of 2 dividing $q^2 - 1$.

First suppose that $\mathbf{C}^{F^*}$ is of type $E_6(q)$ or $A_2(q)$. Then $\bar{\mathbf{C}}_j^{F^*}$ is of type $E_6(q)$ for all j, $2 \leq j \leq 5$. By the formula for the character degrees of unipotent characters of 2-defect 0 of $E_6(q)$, we get that for all j, $2 \leq j \leq 5$,

$$\chi_j(1) = \frac{|G|_{r'}}{3|E_6(q)|(q^2-1)} q^7 \Phi_1^6 \Phi_2^4 \Phi_4^2 \Phi_5 \Phi_8 = \frac{|G|_{r'}}{3|E_6(q)|} q^7 \Phi_1^5 \Phi_2^3 \Phi_4^2 \Phi_5 \Phi_8. \tag{10}$$

Here, as before, we use Φ_d to denote $\Phi_d(q)$.

The unipotent character of defect 2 of $A_2(q)$ is of degree $q(q+1)$ whence

$$\chi_1(1) = \frac{|G|_{r'}}{3|E_6(q)||A_2(q)|}(q+1) q^8 \Phi_1^6 \Phi_2^4 \Phi_4^2 \Phi_5 \Phi_8 = \frac{|G|_{r'}}{3|E_6(q)|} \frac{q^8 \Phi_1^4 \Phi_2^4 \Phi_4^2 \Phi_5 \Phi_8}{\Phi_3^2}. \tag{11}$$

Consider the element $2\chi_1 - \sum_{2\leq j\leq 5} \delta_j \chi_j$ of $L^0(G,b)$ as in Proposition 3.3(ii). By the equation above, all χ_j, $2 \leq j \leq 5$ have the same degree. Hence, by Proposition 3.3(iii), $\delta_j = \delta_i$ for all i, j such that $2 \leq i, j \leq 5$. Thus,

$$2\chi_1(1) - \sum_{2\leq j\leq 5} \delta_j \chi_j(1) = 0$$

implies that

$$q\Phi_2 - 2\delta_2 \Phi_1 \Phi_3^2 = 0,$$

but this is impossible since p_3 does not divide $\Phi_2(q)$.

The case that $\mathbf{C}^{F^*}$ is of type $E_6(q)A_2(q)$ is similar with p_3 replaced by p_6. □

20. Proof of Theorem 1.1

Proof of Theorem 1.1. By Theorem 5.1, and using its notation, it suffices to show that $|\mathrm{Irr}_K(G,b)| = 8$. Arguing inductively, in order to prove Theorem 1.1 we may assume, by Theorem 4.1, that G is a quasi-simple finite group with a centre of odd order. We also may assume, by [48, Theorem 3.7], that b is a non-principal block. If $G/Z(G)$ is a sporadic simple group then by Proposition 6.2 we have $G \cong Co_3$ and by Proposition 6.3, we have $|\mathrm{Irr}_K(G,b)| = 8$. By the results in §§7 and 8, $G/Z(G)$ is neither a finite simple group with an exceptional Schur multiplier nor an alternating group. If $G/Z(G)$ is a finite group of Lie type in characteristic 2 then, by Proposition 9.1, we have $G \cong \mathrm{PSL}_2(8)$ and b is the principal block; in particular, we have again $|\mathrm{Irr}_K(G,b)| = 8$. Let q be an odd prime power and n a positive integer. By Theorems 12.1 and 13.1, the group $G/Z(G)$ cannot be isomorphic to $\mathrm{PSL}_n(q)$ or $\mathrm{PSU}_n(q)$; alternatively, $|\mathrm{Irr}_K(G,b)| = 8$ holds in these cases as a consequence of [6]. If $G/Z(G)$ is isomorphic to one of $\mathrm{PSp}_{2n}(q)$, $(n \geq 2)$, $\mathrm{P}\Omega_{2n+1}(q)$, $(n \geq 3)$, or $\mathrm{P}\Omega_{2n}^{\pm}(q)$, $(n \geq 4)$ then, by Theorem 14.1, the block b is nilpotent; in particular, $|\mathrm{Irr}_K(G,b)| = 8$. By Proposition 15.1, $G/Z(G)$ cannot be isomorphic to a simple group of type $G_2(q)$. Since b is assumed to be non-principal, $G/Z(G)$ cannot be isomorphic to a simple group of type ${}^2G_2(q)$, and by Proposition 15.3, $G/Z(G)$ cannot be isomorphic to a simple group of type ${}^3D_4(q)$. If $G/Z(G)$ is isomorphic to one of the remaining exceptional simple groups $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ then b is Morita equivalent to a block of a finite group L such that $|L/Z(L)|$ is smaller than $|G/Z(G)|$, by Propositions 17.1, 19.2, 19.3, 19.4 and 19.5, respectively. Theorem 1.1 follows inductively. □

Remark 20.1. We do not know whether there are actually any blocks with an elementary abelian defect group of order 8 if $G/Z(G)$ is of one of the exceptional types F or E. If not, one could avoid the rather tedious calculations from §16 onwards.

21. Appendix

We provide a proof for a result announced by Rouquier in [71]. The notation is as in [50, Appendix]. Let $\mathcal{O}$ be a complete local commutative Noetherian ring having an algebraically closed residue field k of characteristic 2; we allow the case $\mathcal{O} = k$. For A, B two symmetric $\mathcal{O}$-algebras, a bounded complex X of A-B-bimodules which are projective as left A-modules and as right B-modules is said to *induce a stable equivalence* if there are isomorphisms of complexes of bimodules $X \otimes_B X^* \cong A \oplus Y$ and $X^* \otimes_A X \cong B \oplus Z$ with Y and Z homotopy equivalent to bounded complexes of projective A-A-bimodules and B-B-bimodules, respectively. If Y and Z are homotopic to zero then X is called a *Rickard complex*.

Theorem 21.1 (cf. [71, Theorem 6.10])**.** *Let G be a finite group, let b be a block of $\mathcal{O}G$ with an elementary abelian defect group of order 8, set $H = N_G(P)$ and denote by c the block of $\mathcal{O}H$ satisfying $\mathrm{Br}_{\Delta P}(b) = \mathrm{Br}_{\Delta P}(c)$. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be source idempotents such that $\mathrm{Br}_{\Delta P}(i) = \mathrm{Br}_{\Delta P}(j)$. There is a bounded complex of $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodules whose components are finite direct sums of summands of the bimodules $\mathcal{O}Gi \otimes_{\mathcal{O}Q} j\mathcal{O}H$, with Q running over the subgroups of P, such that X induces a stable equivalence.*

Proof. The proof follows the lines of [71, 6.3]. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Since P is abelian, the fusion systems on P determined by i and by j are equal to that of $kN(P, e_P)e_P$. Hence, for any subgroup Q of P, we have

$$N_G(Q, e_Q)/C_G(Q) \cong N_H(Q, f_Q)/C_H(Q)$$

and both sides have odd order (either 1 or 3 in case Q is a proper subgroup of P). The blocks e_Q, f_Q lift to unique blocks $\hat{e}_Q$, $\hat{f}_Q$ of $\mathcal{O}C_G(Q)$, $\mathcal{O}C_H(Q)$, respectively. The images of $\hat{e}_Q$, $\hat{f}_Q$ in $\mathcal{O}C_G(Q)/Q$, $\mathcal{O}C_H(Q)/Q$ are blocks, denoted by $\bar{e}_Q$, $\bar{f}_Q$, respectively. Suppose now that Q has order 2. Then $\bar{e}_Q$, $\bar{f}_Q$ have the Klein four group P/Q as defect group, and $C_H(Q)/Q$ is the normaliser in $C_G(Q)/Q$ of P/Q. Thus $\bar{f}_Q$ is in fact the Brauer correspondent of $\bar{e}_Q$. By [24, Theorem 1.1], the source algebras of blocks with a Klein four defect group V_4 are either $\mathcal{O}V_4$, or $\mathcal{O}A_4$, or $\mathcal{O}A_5b_0$, where b_0 is the principal block of $\mathcal{O}A_5$. By [68, §3], there is an explicitly described two-term splendid Rickard complex between $\mathcal{O}A_4$ and $\mathcal{O}A_5b_0$. Thus there is a Rickard complex of $\mathcal{O}C_G(Q)/Q\bar{e}_Q$-$\mathcal{O}C_H(Q)/Q\bar{f}_Q$-bimodules $\bar{C}_Q$ of the form

$$\bar{C}_Q = \cdots \longrightarrow 0 \longrightarrow \bar{N}_Q \xrightarrow{\bar{\Phi}_Q} \bar{e}_Q C_G(Q)/Q\bar{f}_Q \longrightarrow 0 \longrightarrow \cdots$$

for some (possibly zero) projective bimodule $\bar{N}_Q$. By [70, 10.2.11], this complex lifts to a Rickard complex of $\mathcal{O}C_G(Q)\hat{e}_Q$-$\mathcal{O}C_H(Q)\hat{f}_Q$-bimodules of the form

$$C_Q = \cdots \longrightarrow 0 \longrightarrow N_Q \xrightarrow{\Phi_Q} \hat{e}_Q C_G(Q)\hat{f}_Q \longrightarrow 0 \longrightarrow \cdots$$

where N_Q is a projective $\mathcal{O}(C_G(Q) \times \mathcal{O}C_H(Q))/\Delta Q$-module lifting $\bar{N}_Q$, inflated to $\mathcal{O}(C_G(Q) \times C_H(Q))$, and where Φ_Q lifts the map $\bar{\Phi}_Q$. By adapting arguments of Marcuş [55, 5.5] this complex extends to the group

$$T = N_{G\times H}(\Delta Q) \cap (N_G(Q, e_Q) \times N_H(Q, f_Q))$$

and T contains $C_G(Q) \times C_H(Q)$ as normal subgroup of odd index at most 3 by the above remarks. One can see this also directly: the modules of the complex C_Q are clearly T-stable, hence extend to T (cf. [70, 10.2.13]), and the map $\bar{\Phi}$ lifts to a T-stable map Ψ because N_Q remains projective when considered as $\mathcal{O}T/\Delta Q$-module. We set

$$V_Q = \mathrm{Ind}_T^{G\times H}(N_Q)\ .$$

The inclusion $C_G(Q) \subseteq G$ induces an $\mathcal{O}T$-homomorphism $e_Q\mathcal{O}C_G(Q)f_Q \longrightarrow b\mathcal{O}Gc$, which, by adjunction, yields a homomorphism of $\mathcal{O}(G\times H)$-modules

$$\alpha_Q : \mathrm{Ind}_T^{G\times H}(e_Q\mathcal{O}C_G(Q)f_Q) \longrightarrow b\mathcal{O}Gc\ .$$

Set $\psi_Q = \alpha_Q \circ \mathrm{Ind}_T^{G\times H}(\Phi_Q) : V_Q \to b\mathcal{O}Gc$ and define the complex X by

$$X = \ \cdots \longrightarrow 0 \longrightarrow \oplus_Q V_Q \xrightarrow{\ \oplus_Q \Psi_Q\ } b\mathcal{O}Gc \longrightarrow 0 \longrightarrow \cdots$$

with $b\mathcal{O}Gc$ in degree zero, where Q runs over a set of representatives of the $N_G(P, e_P)$-conjugacy classes of subgroups of order 2 of P. One checks that if Q, R are two subgroups of order 2 which are not $N_G(P, e_P)$-conjugate then $V_R(\Delta Q) = \{0\}$. This implies that $e_Q X(\Delta Q) f_Q \ \simeq C_Q \otimes_{\mathcal{O}} k$, and this is a Rickard complex of $kC_G(Q)e_Q$-$C_H(Q)f_Q$-bimodules. Moreover, $b\mathcal{O}Gc$ is a direct summand of $\mathcal{O}Gi \otimes_{\mathcal{O}P} j\mathcal{O}H$, and V_Q is a direct sum of summands of $\mathcal{O}Gi \otimes_{\mathcal{O}Q} j\mathcal{O}H$ because it is obtained from lifting, inflating and inducing a projective bimodule. Thus another result of Rouquier (with a proof given in [50, Theorem A.1]) applies, showing that X induces a stable equivalence. $\square$

Remark 21.2. We have used [24, Theorem 1.1] for the description of Rickard complexes for blocks with a Klein four group, which requires the classification of finite simple groups. One can avoid this by making use of another technique of Rickard, replacing one of the endo-permutation sources of a simple module by a p-permutation resolution (see e.g. [49, Theorem 1.3]). The only difference this makes is that the resulting complex may have more than two non-zero components. One can be more precise in Theorem 21.1 if E is either trivial or has order 7. If E is trivial, b is nilpotent, and its source algebra is of the form $\mathrm{End}_{\mathcal{O}}(V) \otimes_{\mathcal{O}} \mathcal{O}P$ for some indecomposable endo-permutation $\mathcal{O}P$-module V with P as vertex. Using again [24, Theorem 1.1] we get that $V \cong \Omega_P^n(\mathcal{O})$ for some integer n. Equivalently, if E is trivial then $\mathcal{O}Gb$ is Morita equivalent to $\mathcal{O}P$ via an $\mathcal{O}Gb$-$\mathcal{O}P$-bimodule which, when viewed as $\mathcal{O}(G\times P)$-module, has vertex $\Delta P = \{(u,u) \mid u \in P\}$ and source $\Omega^n_{\Delta P}(\mathcal{O})$ for some integer n. By applying $\Omega^{-n}_{\mathcal{O}(G\times P)}$ to M one gets a stable equivalence of Morita type given by a bimodule with vertex ΔP and trivial source. Similarly, if E is cyclic of order 7, then by a result of Puig [64], there is a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}P$ given by a bimodule with vertex ΔP and trivial source.

Acknowledgements. The authors would like to thank Felix Noeske for bringing to their attention the work in [60] and Gunter Malle for pointing the authors to the references [5], [6] and for his careful reading of parts of the manuscript. The authors would further like to thank the referee for many valuable comments. The second author made several visits to the University of Aberdeen in 2008 and 2009 during which parts of this work were carried out. He is grateful to the Institute of Mathematics, University of Aberdeen for its kind hospitality. For this research the second author was partially supported by the Japan Society for Promotion of Science (JSPS), Grant-in-Aid for Scientific Research (C)20540008, 2008–2010.

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